Simplifying Probabilistic Programs Using Computer Algebra

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joint work with Ken and others at IU and McMaster

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$$\mathrm{Msum}\big(\mathrm{Weight}(1/2, \mathrm{Ret}(\mathrm{true})), \mathrm{Weight}(1/2, \mathrm{Ret}(\mathrm{false}))\big)$$

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$$\mathrm{Gaussian}(0,\sqrt{2})$$

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$$\mathrm{Weight}\left(\frac{\exp\left(-\frac{y^2}{4}\right)}{2\cdot\sqrt{\pi}}, \mathrm{Gaussian}\left(\frac{y}{2}, \frac{1}{\sqrt{2}}\right)\right)$$

Embedding our language in Maple

$$\frac{a : \mathbb{R} \quad b : \mathbb{R}}{\mathsf{Uniform}(a, b) : \mathbb{MR}} \qquad \frac{\mu : \mathbb{R} \quad \sigma : \mathbb{R}^+}{\mathsf{Gaussian}(\mu, \sigma) : \mathbb{MR}} \qquad \frac{\mu : \mathbb{R} \quad \gamma : \mathbb{R}^+}{\mathsf{Cauchy}(\mu, \gamma) : \mathbb{MR}}$$

$$\frac{\nu : \mathbb{R} \quad \mu : \mathbb{R} \quad \gamma : \mathbb{R}^+}{\mathsf{StudentT}(\nu, \mu, \gamma) : \mathbb{MR}} \qquad \frac{\alpha : \mathbb{R}^+ \quad \beta : \mathbb{R}^+}{\mathsf{Beta}(\alpha, \beta) : \mathbb{MR}^+} \qquad \frac{k : \mathbb{R}^+ \quad \theta : \mathbb{R}^+}{\mathsf{Gamma}(k, \theta) : \mathbb{MR}^+}$$

$$\frac{e : A}{\mathsf{Ret}(e) : \mathbb{M}A} \qquad \frac{m : \mathbb{M}A \quad \begin{matrix}[x : A] \\ \vdots \\ m' : \mathbb{M}B\end{matrix}}{\mathsf{Bind}(m, x, m') : \mathbb{M}B} \qquad \frac{\forall i \le n,\ m_i : \mathbb{M}A}{\mathsf{Msum}(m_1, \ldots, m_n) : \mathbb{M}A}$$

$$\frac{e : \mathbb{R}^+ \quad m : \mathbb{M}A}{\mathsf{Weight}(e, m) : \mathbb{M}A} \qquad \frac{e : \mathbb{B} \quad m : \mathbb{M}A \quad m' : \mathbb{M}A}{\mathsf{If}(e, m, m') : \mathbb{M}A} \qquad \frac{\begin{matrix}[\hbar : A \to \mathbb{R}^+] \\ \vdots \\ g : \mathbb{R}^+\end{matrix}}{\mathsf{LO}(\hbar, g) : \mathbb{M}A}$$

Figure: Informal typing rules for how we represent measures

Denotational semantics – abstract and concrete integration

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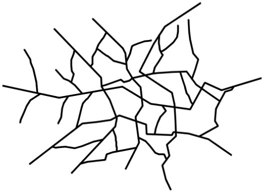

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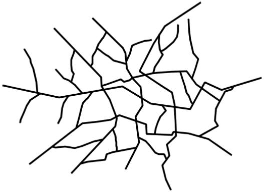

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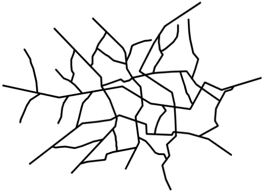

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$$\mathbb{M}A \qquad \simeq \qquad (A \multimapinv \mathbb{R}) \multimapinv \mathbb{R}$$

The plan

1. View the given measure term as a linear operator, which specifies the abstract integral of an arbitrary integrand.

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3. Read off a new measure term from the improved linear operator.

$$(A \multimapinv \mathbb{R}) \multimapinv \mathbb{R} \mapsto \mathbb{M}A$$

Measure to linear operator

$$
\begin{aligned}
\mathrm{integ}\big(m, \quad & h\big) = \textstyle\int_{\mathcal{L}(m)}^{\mathcal{U}(m)} \mathcal{D}(m,x)\cdot h(x)\,dx \\
& \quad \text{where } m \text{ is a primitive measure term} \\
& \quad \text{and } x \text{ is fresh} \\
\mathrm{integ}\big(\mathrm{Ret}(e), \quad & h\big) = h(e) \\
\mathrm{integ}\big(\mathrm{Bind}(m,x,m'), \quad & h\big) = \mathrm{integ}\big(m, \lambda x.\, \mathrm{integ}(m', h)\big) \\
\mathrm{integ}\big(\mathrm{Msum}(m,\ldots), \quad & h\big) = \mathrm{integ}(m,h) + \cdots \\
\mathrm{integ}\big(\mathrm{Weight}(e,m), \quad & h\big) = e\cdot \mathrm{integ}(m,h) \\
\mathrm{integ}\big(\mathrm{If}(e,m,m'), \quad & h\big) = \mathrm{If}\big(e, \mathrm{integ}(m,h), \mathrm{integ}(m',h)\big) \\
\mathrm{integ}\big(\mathrm{LO}(\hbar, g), \quad & h\big) = g\{\hbar \mapsto h\} \\
\mathrm{integ}\big(m, \quad & h\big) = \mathrm{Integrate}(m,h) \qquad \text{otherwise}
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patent linearity. $\sin(h(a))^2 + \cos(h(a))^2 - 1$

Density

Measure term m	Bounds $\mathcal{L}(m)$	$\mathcal{U}(m)$	Density $\mathcal{D}(m, x)$
$\text{Uniform}(a, b)$	a	b	$\dfrac{1}{b-a}$
$\text{Gaussian}(\mu, \sigma)$	$-\infty$	$+\infty$	$\dfrac{\exp(-\frac{1}{2} \cdot (\frac{x-\mu}{\sigma})^2)}{\sqrt{2 \cdot \pi} \cdot \sigma}$
$\text{Cauchy}(\mu, \gamma)$	$-\infty$	$+\infty$	$\dfrac{1}{\pi \cdot \gamma}\left(1 + (\dfrac{x-\mu}{\gamma})^2\right)^{-1}$
$\text{StudentT}(\nu, \mu, \gamma)$	$-\infty$	$+\infty$	$\dfrac{\left(1 + (\frac{x-\mu}{\gamma})^2 \nu\right)^{-(\nu+1)/2}}{\frac{\Gamma(\nu/2)}{\Gamma((\nu+1)/2)} \cdot \sqrt{\pi \cdot \nu} \cdot \gamma}$
$\text{Beta}(\alpha, \beta)$	0	1	$\dfrac{x^{\alpha-1} \cdot (1-x)^{\beta-1}}{\mathrm{B}(\alpha, \beta)}$
$\text{Gamma}(k, \theta)$	0	$+\infty$	$\dfrac{x^{k-1} \cdot \exp(-\frac{x}{\theta})}{\Gamma(k) \cdot \theta^k}$

linear operator back to measure

$$
\begin{aligned}
\mathrm{toM}\Big(\hbar, \textstyle\int_a^b g\,dx, \quad c\Big) &= \mathrm{weight}\Big(e_1, \mathrm{bind}(m, x, \mathrm{weight}(e_2, m'))\Big) \\
&\quad \text{where } x \neq \hbar \text{ and } (e, m') = \mathrm{unweight}\Big(\mathrm{toM}(\hbar, g, c \wedge a < x < b)\Big) \\
&\quad (m, e') = \mathrm{recognize}(e, x, a, b, c) \\
&\quad e_1 \cdot e_2 = e' \text{ where } e_1 \text{ does not contain } x \text{ free} \\
\mathrm{toM}\Big(\hbar, \hbar(e), \quad c\Big) &= \mathrm{Ret}(e) \\
\mathrm{toM}\Big(\hbar, g + \cdots, \quad c\Big) &= \mathrm{Msum}\Big(\mathrm{toM}(\hbar, g, c), \ldots\Big) \\
\mathrm{toM}\Big(\hbar, e \cdot g, \quad c\Big) &= \mathrm{weight}\Big(e, \mathrm{toM}(\hbar, g, c)\Big) \text{where } e \text{ does not contain } \hbar \text{ free} \\
\mathrm{toM}\Big(\hbar, \mathrm{If}(e, g, g'), \quad c\Big) &= \mathrm{If}\Big(e, \mathrm{toM}(\hbar, g, c \wedge e), \mathrm{toM}(\hbar, g', c \wedge \neg e)\Big) \\
\mathrm{toM}\Big(\hbar, \mathrm{Integrate}(m, h), c\Big) &= \mathrm{bind}\Big(m, x, \mathrm{toM}(\hbar, h(x), c)\Big) \qquad \text{where } x \text{ is fresh} \\
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$$\begin{aligned}
\mathrm{bind}(m, \quad x, \mathrm{Ret}(x)) &= m \\
\mathrm{bind}(\mathrm{Ret}(e), x, m) &= m\{x \mapsto e\} \\
\mathrm{bind}(m, \quad x, m') &= \mathrm{Bind}(m, x, m') \quad \text{otherwise}
\end{aligned}$$

$$\begin{aligned}
\mathrm{weight}(1, m) &= m \\
\mathrm{weight}(0, m) &= \mathrm{Msum}() \\
\mathrm{weight}(e, \mathrm{Weight}(e', m)) &= \mathrm{weight}(e \cdot e', m) \\
\mathrm{weight}(e, m) &= \mathrm{Weight}(e, m) \text{ otherwise}
\end{aligned}$$

$$\begin{aligned}
\mathrm{unweight}\Big(\mathrm{Weight}(e, m)\Big) &= \Big(e, m\Big) \\
\mathrm{unweight}\Big(\mathrm{Msum}(m_1, \ldots, m_n)\Big) &= \Big(e, \mathrm{Msum}(\mathrm{weight}(\tfrac{1}{e}, m_1), \ldots, \mathrm{weight}(\tfrac{1}{e}, m_n))\Big) \\
&\quad \text{where } e = e_1 + \cdots + e_n \text{ and } (e_i, _) = \mathrm{unweight}(m_i) \\
\mathrm{unweight}\Big(m\Big) &= \Big(1, m\Big) \qquad \text{otherwise}
\end{aligned}$$

Density recognition

$$\mathrm{Weight}\left(\frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\cdot\pi}}\cdot\frac{\exp\left(-\frac{(y-x)^2}{2}\right)}{\sqrt{2\cdot\pi}},\ \mathrm{Ret}(x)\right)$$

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DEs FTW

Density recognition

Measure term m	Density $\mathcal{D}(m, x)$	Holonomic representation $\frac{p_0(x)}{p_1(x)} = -\frac{\frac{d}{dx}\mathcal{D}(m,x)}{\mathcal{D}(m,x)}$
$\text{Uniform}(a, b)$	$\frac{1}{b-a}$	0
$\text{Gaussian}(\mu, \sigma)$	$\frac{\exp(-\frac{1}{2}\cdot(\frac{x-\mu}{\sigma})^2)}{\sqrt{2\cdot\pi}\cdot\sigma}$	$\frac{x-\mu}{\sigma^2}$
$\text{Cauchy}(\mu, \gamma)$	$\frac{\left(1+(\frac{x-\mu}{\gamma})^2\right)^{-1}}{\pi\cdot\gamma}$	$\frac{2\cdot(x-\mu)}{(x-\mu)^2+\gamma^2}$
$\text{StudentT}(\nu, \mu, \gamma)$	$\frac{\left(1+(\frac{x-\mu}{\gamma})^2\nu\right)^{-(\nu+1)/2}}{\frac{\Gamma(\nu/2)}{\Gamma((\nu+1)/2)}\cdot\sqrt{\pi\cdot\nu}\cdot\gamma}$	$\frac{(\nu+1)\cdot(x-\mu)}{(x-\mu)^2+\gamma^2\cdot\nu}$
$\text{Beta}(\alpha, \beta)$	$\frac{x^{\alpha-1}\cdot(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)}$	$\frac{(\alpha+\beta-2)\cdot x-(\alpha-1)}{x\cdot(1-x)}$
$\text{Gamma}(k, \theta)$	$\frac{x^{k-1}\cdot\exp(-\frac{x}{\theta})}{\Gamma(k)\cdot\theta^k}$	$\frac{\frac{x}{\theta}+1-k}{x}$

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Improving linear operators algebraically

- The good (automatic and efficient)
  - $+$ and $\cdot$ are commutative and associative, with identities 0 and 1,
  - $g_1 + (g_1 + g_2) = 2 \cdot g_1 + g_2$,
  - $\mathbb{Q}$ arithmetic (so $\frac{1}{1-0}x = x$).

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  - `value`, `simplify`

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- The ugly (but rather effective)
  - Follow the grammar of patently linear expressions (not "just" syntax)
  - Keep control flow intact (i.e. when sub-expressions must be patently linear)
  - Push integrals "inwards" when $\hbar$ does not depend on integ. var.
  - Reduce integral bounds by pulling domain information from If

Two human-useful program transformations

- Banish n levels. Example for 1 level:

$$\int_{-\infty}^{\infty} \int_{0}^{1} \int_{-2}^{2} f(w, z, y) h(z, w) dy\, dw\, dz$$

$$\mapsto$$

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- Reparametrization:

$$\int_a^b g(x)h(f(x))dx \mapsto \int_{f(a)}^{f(b)} \frac{g(f^{-1}(z))h(z)}{f'(f^{-1}(z))}dz$$

assuming f is differentiable and monotone increasing.

Next steps

- Arrays:

$$\text{Bind}(\text{Gaussian}(0,1), x,$$
$$\text{Bind}(\mathit{Plate}(\mathit{ary}(n, i, \text{Weight}(\mathcal{D}(\text{Gaussian}(x,1), t_i), \text{Ret}(\mathit{Unit})))), \mathit{ys},$$
$$\text{Ret}(x)))$$

$$\mapsto$$

$$\text{Weight}\left(\frac{e^{\frac{(\Sigma_{i=1}^{n} t_i)^2 - n\Sigma_{i=1}^{n} t_i^2 - \Sigma_{i=1}^{n} t_i^2}{2n+2}}}{\sqrt{2^{n+1}\pi^n(2+2n)}}, \text{Gaussian}\left(\frac{1}{n+1}\Sigma_{i=1}^{n} t_i, \frac{1}{\sqrt{n+1}}\right)\right)$$

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- Domain restriction arithmetic

$$\int_{-\infty}^{\infty} \begin{cases} 0 & x < a \\ 1 & \text{otherwise} \end{cases} \cdot \begin{cases} -1 & x < a+5 \\ 0 & \text{otherwise} \end{cases} h(x)\, dx \mapsto \int_{a}^{a+5} -h(x)\, dx$$

Hakaru

Play with it: https://github.com/hakaru-dev/hakaru