Symbolic Bayesian inference by lazy partial evaluation

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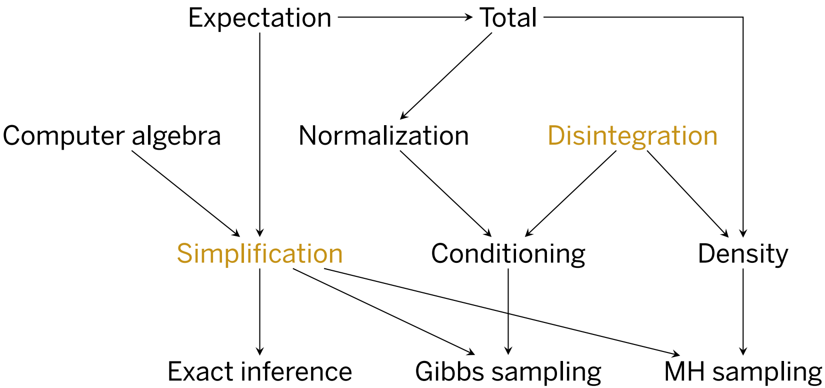
Program transformations galore



Disintegration for medical diagnosis

Diseases A and B are equally prevalent.

Disease A causes one of symptoms 1, 2, 3 with equal probability.

Disease B causes one of symptoms 1, 2 with equal probability.

$$\wr A \mapsto 1/2, B \mapsto 1/2 \int \qquad : \mathbb{M}\ \text{Disease}$$

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$$
\begin{array}{l}
\textbf{do}\ \{\mathit{disease} \leftarrow \lbag A \mapsto 1/2, B \mapsto 1/2 \rbag\,;\\
\qquad \mathit{symptom} \leftarrow \textbf{case}\ \mathit{disease}\ \textbf{of}\\
\qquad\qquad\qquad\qquad A \to \lbag 1 \mapsto 1/3, 2 \mapsto 1/3, 3 \mapsto 1/3 \rbag\\
\qquad\qquad\qquad\qquad B \to \lbag 1 \mapsto 1/2, 2 \mapsto 1/2 \rbag;\\
\qquad \textbf{return}\ (\mathit{symptom}, \mathit{disease})\} \qquad : \mathbb{M}\,(\text{Symptom} \times \text{Disease})
\end{array}
$$

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&\qquad \textbf{return}\ (\mathit{symptom}, \mathit{disease})\} \qquad : \mathbb{M}\ (\text{Symptom} \times \text{Disease})
\end{aligned}
$$

$$
\begin{aligned}
= \wr &(A,1) \mapsto 1/6, (A,2) \mapsto 1/6, (A,3) \mapsto 1/6,\\
&(B,1) \mapsto 1/4, (B,2) \mapsto 1/4 \wr
\end{aligned}
$$

A	1/6	1/6	1/6
B	1/4	1/4	0
	1	2	3

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$$
\begin{array}{l}
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\qquad \textbf{return}\ (\mathit{symptom}, \mathit{disease})\} \qquad : \mathbb{M}\,(\mathrm{Symptom} \times \mathrm{Disease})
\end{array}
$$

$$
\begin{array}{l}
= \wr (A,1) \mapsto 1/6, (A,2) \mapsto 1/6, (A,3) \mapsto 1/6, \\
\quad (B,1) \mapsto 1/4, (B,2) \mapsto 1/4 \int
\end{array}
$$

$$
\begin{array}{l}
\Rightarrow \lambda \mathit{symptom}.\ \textbf{case}\ \mathit{symptom}\ \textbf{of} \\
\qquad\qquad 1 \to \wr A \mapsto 1/6, B \mapsto 1/4 \int \\
\qquad\qquad 2 \to \wr A \mapsto 1/6, B \mapsto 1/4 \int \\
\qquad\qquad 3 \to \wr A \mapsto 1/6 \int \\
\quad : \mathrm{Symptom} \to \mathbb{M}\,\mathrm{Disease}
\end{array}
$$

A	1/6	1/6	1/6
B	1/4	1/4	0
	1	2	3

Disintegration on a zero-probability observation

Diseases A and B are equally prevalent.

Disease A causes a symptom chosen uniformly from $[0, 3] \subset \mathbb{R}$.

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$$
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&\textbf{do}\ \{\mathit{disease} \leftsquigarrow \llparenthesis A \mapsto {}^{1}\!/_{2}, B \mapsto {}^{1}\!/_{2} \rrparenthesis\,; \\
&\qquad \mathit{symptom} \leftsquigarrow \textbf{case}\ \mathit{disease}\ \textbf{of} \\
&\qquad\qquad\qquad\quad A \to \textbf{uniform}\ 0\ 3 \\
&\qquad\qquad\qquad\quad B \to \textbf{uniform}\ 0\ 2; \\
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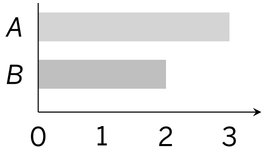

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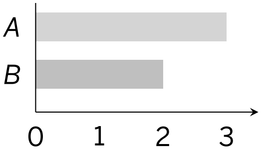
\end{array}
$$

$$
\begin{array}{l}
\Rightarrow \lambda \mathit{symptom}.\ \textbf{if}\ \mathit{symptom} \leq 2\\
\qquad\qquad\qquad \textbf{then}\ \lbag A \mapsto 1/6, B \mapsto 1/4 \rbag\\
\qquad\qquad\qquad \textbf{else}\ \ \lbag A \mapsto 1/6 \rbag\\
\quad : \text{Symptom} \to \mathbb{M}\ \text{Disease}
\end{array}
$$



Disintegration on a zero-probability observation

Choose *disease* uniformly from $[1, 3] \subset \mathbb{R}$.

Choose *symptom* uniformly from $[0, \mathit{disease}] \subset \mathbb{R}$.

$$\textbf{uniform}\ 1\ 3 \qquad : \mathbb{M}\ \text{Disease}$$

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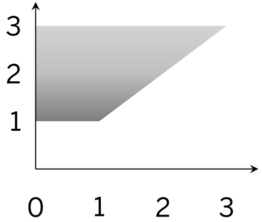

Disintegration on a zero-probability observation

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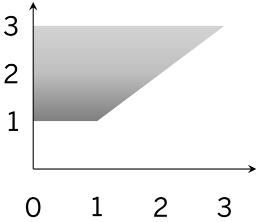
$$\begin{aligned}&\textbf{do}\ \{\mathit{disease} \leftsquigarrow \textbf{uniform}\ 1\ 3;\\ &\qquad \mathit{symptom} \leftsquigarrow \textbf{uniform}\ 0\ \mathit{disease};\\ &\qquad \textbf{return}\ (\mathit{symptom}, \mathit{disease})\} \qquad : \mathbb{M}\ (\mathrm{Symptom} \times \mathrm{Disease})\end{aligned}$$

$$\begin{aligned}\Rightarrow \lambda \mathit{symptom}.\ &\textbf{do}\ \{\mathit{disease} \leftsquigarrow \textbf{uniform}\ 1\ 3;\\ &\qquad \textbf{if}\ 0 \le \mathit{symptom} \le \mathit{disease}\\ &\qquad\quad \textbf{then}\ \wr \mathit{disease} \mapsto {}^{1}/_{\mathit{disease}} \int\\ &\qquad\quad \textbf{else}\ \ \wr \int\}\\ &: \mathrm{Symptom} \to \mathbb{M}\ \mathrm{Disease}\end{aligned}$$



Measure semantics

$$[\![\mathbb{M}\ \alpha]\!] \quad = \quad (\alpha \to \mathbb{R}) \to \mathbb{R}$$

Measure semantics

$$
\begin{aligned}
[\![\mathbb{M}\ \alpha]\!] &= (\alpha \to \mathbb{R}) \to \mathbb{R} \\
[\![\wr A \mapsto 1/2, B \mapsto 1/2 \int]\!] &= \lambda c.\, \frac{c(A)}{2} + \frac{c(B)}{2} \\
[\![\textbf{return}\ (\mathit{symptom}, \mathit{disease})]\!] &= \lambda c.\, c(\mathit{symptom}, \mathit{disease}) \\
&= [\![\wr(\mathit{symptom}, \mathit{disease}) \mapsto 1 \int]\!] \\
[\![\textbf{uniform}\ 1\ 3]\!] &= \lambda c.\, \int_1^3 \frac{c(x)}{2}\, \mathrm{d}x \\
[\![\textbf{lebesgue}]\!] &= \lambda c.\, \int_{-\infty}^{\infty} c(x)\, \mathrm{d}x \\
[\![\textbf{do}\ \{x \leftsquigarrow m;\ k\}]\!] &= \lambda c.\, [\![m]\!](\lambda x.\, [\![k]\!]c)
\end{aligned}
$$

Measure semantics

$$
\begin{aligned}
[\![\mathbb{M}\ \alpha]\!] &= (\alpha \to \mathbb{R}) \to \mathbb{R} \\[1em]
[\![\wr A \mapsto {}^1\!/\!_2, B \mapsto {}^1\!/\!_2 \int]\!] &= \lambda c.\, \frac{c(A)}{2} + \frac{c(B)}{2} \\
[\![\textbf{return}\ (\mathit{symptom}, \mathit{disease})]\!] &= \lambda c.\, c(\mathit{symptom}, \mathit{disease}) \\
&= [\![\wr (\mathit{symptom}, \mathit{disease}) \mapsto 1 \int]\!] \\[1em]
[\![\textbf{uniform}\ 1\ 3]\!] &= \lambda c.\, \int_1^3 \frac{c(x)}{2}\, dx \\
[\![\textbf{lebesgue}]\!] &= \lambda c.\, \int_{-\infty}^{\infty} c(x)\, dx \\
[\![\textbf{do}\ \{x \leftsquigarrow m;\ k\}]\!] &= \lambda c.\, [\![m]\!](\lambda x.\, [\![k]\!] c) \\
\left[\!\!\left[\begin{array}{l}\textbf{do}\ \{d \leftsquigarrow \textbf{uniform}\ 1\ 3; \\ \quad\ \ s \leftsquigarrow \textbf{uniform}\ 0\ d; \\ \quad\ \ \textbf{return}\ (s, d)\}\end{array}\right]\!\!\right] &= \lambda c.\, \int_1^3 \int_0^d \frac{c(s, d)}{2 \cdot d}\, ds\, dd
\end{aligned}
$$

Disintegration specification

$$[\![m]\!] = [\![\mathbf{do}\ \{s \leftsquigarrow \mathbf{lebesgue};\ d \leftsquigarrow k;\ \mathbf{return}\ (s, d)\}]\!]$$

Disintegration specification

$$[\![m]\!] = [\![\mathbf{do}\ \{s \leftsquigarrow \mathbf{lebesgue};\ d \leftsquigarrow k;\ \mathbf{return}\ (s,d)\}]\!]$$

$$\begin{aligned} m = \mathbf{do}\ \{&d \leftsquigarrow \mathbf{uniform}\ 1\ 3; \\ &s \leftsquigarrow \mathbf{uniform}\ 0\ d; \\ &\mathbf{return}\ (s,d)\} \end{aligned}$$

$$\begin{aligned} k = \mathbf{do}\ \{&d \leftsquigarrow \mathbf{uniform}\ 1\ 3; \\ &\mathbf{if}\ 0 \le s \le d \\ &\quad \mathbf{then}\ \wr d \mapsto {}^1/_d \int \\ &\quad \mathbf{else}\ \ \wr\int\} \end{aligned}$$

Disintegration specification

$$\llbracket m \rrbracket = \llbracket \textbf{do } \{ s \leftsquigarrow \textbf{lebesgue};\ d \leftsquigarrow k;\ \textbf{return } (s,d) \} \rrbracket$$

$$m = \textbf{do } \{ d \leftsquigarrow \textbf{uniform } 1\ 3;\quad s \leftsquigarrow \textbf{uniform } 0\ d;\quad \textbf{return } (s,d) \}$$

$$k = \textbf{do } \{ d \leftsquigarrow \textbf{uniform } 1\ 3;\quad \textbf{if } 0 \le s \le d \ \textbf{then } \lbag d \mapsto 1/d \rbag \ \textbf{else } \lbag \rbag \}$$

$$\llbracket k \rrbracket = \lambda c.\ \int_1^3 \frac{\textbf{if } 0 \le s \le d \textbf{ then } \frac{c(d)}{d} \textbf{ else } 0}{2}\, \mathrm{d}d$$

$$\llbracket \textbf{do } \{ s \leftsquigarrow \textbf{lebesgue};\ d \leftsquigarrow k;\ \textbf{return } (s,d) \} \rrbracket$$

$$= \lambda c.\ \int_{-\infty}^{\infty} \int_1^3 \frac{\textbf{if } 0 \le s \le d \textbf{ then } \frac{c(s,d)}{d} \textbf{ else } 0}{2}\, \mathrm{d}d\, \mathrm{d}s$$

$$= \lambda c.\ \int_1^3 \int_0^d \frac{c(s,d)}{2 \cdot d}\, \mathrm{d}s\, \mathrm{d}d = \llbracket m \rrbracket$$

Useful but unspecified and thus unautomated before

1. Initialise $x_{0,1:n}$.
2. For $i = 0$ to $N-1$
   - Sample $x_1^{(i+1)} \sim p(x_1 | x_2^{(i)}, x_3^{(i)}, \ldots, x_n^{(i)})$.
   - Sample $x_2^{(i+1)} \sim p(x_2 | x_1^{(i+1)}, x_3^{(i)}, \ldots, x_n^{(i)})$.

     $\vdots$
   - Sample $x_j^{(i+1)} \sim p(x_j | x_1^{(i+1)}, \ldots, x_{j-1}^{(i+1)}, x_{j+1}^{(i)}, \ldots, x_n^{(i)})$.

     $\vdots$
   - Sample $x_n^{(i+1)} \sim p(x_n | x_1^{(i+1)}, x_2^{(i+1)}, \ldots x_{n-1}^{(i+1)})$.

Figure 12. Gibbs sampler.

Borel paradox

Radio Yerevan

Question: Is it correct that Grigori Grigorievich Grigoriev won a luxury car at the All-Union Championship in Moscow?

Answer: In principle, yes.

But first of all it was not Grigori Grigorievich Grigoriev, but Vassili Vassilievich Vassiliev.

Second, it was not at the All-Union Championship in Moscow, but at a Collective Farm Sports Festival in Smolensk.

Third, it was not a car, but a bicycle.

And fourth he didn’t win it, but rather it was stolen from him.

Automatic disintegrator

Question: Is it correct that our disintegrator is a lazy evaluator?

Answer: In principle, yes.

$$\text{evaluate} : \lceil \alpha \rceil \to \mathit{H} \to (\alpha \times \mathit{H})$$

Automatic disintegrator

Question: Is it correct that our disintegrator is a lazy evaluator?

Answer: In principle, yes.

But first of all it is not an evaluator, but a partial evaluator.

$$\text{evaluate} : \lceil \alpha \rceil \to H \to (\lfloor \alpha \rfloor \times H)$$

Automatic disintegrator

Question: Is it correct that our disintegrator is a lazy evaluator?

Answer: In principle, yes.

But first of all it is not an evaluator, but a partial evaluator.

Second, it not only evaluates terms, but also performs random choices.

$$\text{evaluate} : \lceil \quad \alpha \rceil \to H \to (\lfloor \alpha \rfloor \to H \to \lfloor \mathbb{M}\ \gamma \rfloor) \to \lfloor \mathbb{M}\ \gamma \rfloor$$

$$\text{perform} : \lceil \mathbb{M}\ \alpha \rceil \to H \to (\lfloor \alpha \rfloor \to H \to \lfloor \mathbb{M}\ \gamma \rfloor) \to \lfloor \mathbb{M}\ \gamma \rfloor$$

Automatic disintegrator

Question: Is it correct that our disintegrator is a lazy evaluator?

Answer: In principle, yes.

But first of all it is not an evaluator, but a partial evaluator.

Second, it not only evaluates terms, but also performs random choices.

Third, it not only produces outcomes and values, but also constrains them.

$$
\begin{aligned}
\text{evaluate} &: \lceil\phantom{\mathbb{M}}\ \alpha\rceil \to H \to (\lfloor\alpha\rfloor \to H \to \lfloor\mathbb{M}\ \gamma\rfloor) \to \lfloor\mathbb{M}\ \gamma\rfloor \\
\text{perform} &: \lceil\mathbb{M}\ \alpha\rceil \to H \to (\lfloor\alpha\rfloor \to H \to \lfloor\mathbb{M}\ \gamma\rfloor) \to \lfloor\mathbb{M}\ \gamma\rfloor \\
\text{constrain-value} &: \lceil\phantom{\mathbb{M}}\ \alpha\rceil \to \lfloor\alpha\rfloor \to H \to (H \to \lfloor\mathbb{M}\ \gamma\rfloor) \to \lfloor\mathbb{M}\ \gamma\rfloor \\
\text{constrain-outcome} &: \lceil\mathbb{M}\ \alpha\rceil \to \lfloor\alpha\rfloor \to H \to (H \to \lfloor\mathbb{M}\ \gamma\rfloor) \to \lfloor\mathbb{M}\ \gamma\rfloor
\end{aligned}
$$

Automatic disintegrator

Question: Is it correct that our disintegrator is a lazy evaluator?

Answer: In principle, yes.

But first of all it is not an evaluator, but a partial evaluator.

Second, it not only evaluates terms, but also performs random choices.

Third, it not only produces outcomes and values, but also constrains them.

And fourth it doesn't produce one term, but searches for a random variable to constrain.

$$
\begin{aligned}
\text{evaluate} &: \lceil \quad \alpha \rceil \to H \to (\lfloor \alpha \rfloor \to H \to \{\lfloor \mathbb{M}\ \gamma \rfloor\}) \to \{\lfloor \mathbb{M}\ \gamma \rfloor\} \\
\text{perform} &: \lceil \mathbb{M}\ \alpha \rceil \to H \to (\lfloor \alpha \rfloor \to H \to \{\lfloor \mathbb{M}\ \gamma \rfloor\}) \to \{\lfloor \mathbb{M}\ \gamma \rfloor\} \\
\text{constrain-value} &: \lceil \quad \alpha \rceil \to \lfloor \alpha \rfloor \to H \to (H \to \{\lfloor \mathbb{M}\ \gamma \rfloor\}) \to \{\lfloor \mathbb{M}\ \gamma \rfloor\} \\
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\end{aligned}
$$

Automatic disintegrator in action

$$\Big\lceil \text{perform}\,(\mathbf{do}\ \{d \leftsquigarrow \mathbf{uniform}\ 1\ 3;\ s \leftsquigarrow \mathbf{uniform}\ 0\ d;\ \mathbf{return}\ (s,d)\})^{[]}$$

$$[d' \leftsquigarrow \mathbf{uniform}\ 1\ 3]$$

$$\text{perform}\,(\mathbf{do}\ \{s \leftsquigarrow \mathbf{uniform}\ 0\ d';\ \mathbf{return}\ (s,d')\})$$

$$[d' \leftsquigarrow \mathbf{uniform}\ 1\ 3;\ s' \leftsquigarrow \mathbf{uniform}\ 0\ d']$$

$$\text{perform}\,(\mathbf{return}\ (s',d'))$$

$$\text{evaluate}\ (s',d') \Rightarrow (s',d')$$

$$\Big\lceil \text{constrain-value}\ s'\ s$$

$$\Big\lceil \text{constrain-outcome}\ (\mathbf{uniform}\ 0\ d')\ s$$

Automatic disintegrator in action

perform (**do** $\{d \leftsquigarrow$ **uniform** $1\ 3$; $s \leftsquigarrow$ **uniform** $0\ d$; **return** $(s, d)\})$ $[]$

$[d' \leftsquigarrow$ **uniform** $1\ 3]$

perform (**do** $\{s \leftsquigarrow$ **uniform** $0\ d'$; **return** $(s, d')\})$

$[d' \leftsquigarrow$ **uniform** $1\ 3$; $s' \leftsquigarrow$ **uniform** $0\ d']$

perform (**return** (s', d'))

evaluate $(s', d') \Rightarrow (s', d')$

constrain-value $s'\ s$

constrain-outcome (**uniform** $0\ d'$) s

nondeterminism

evaluate $0 \Rightarrow 0$

evaluate d'

perform (**uniform** $1\ 3$)

$\Rightarrow d''$

$\Rightarrow d''$

do $\{d'' \leftsquigarrow$ **uniform** $1\ 3$; $\square\}$

$[$**let** $d' = d''$; $s' \leftsquigarrow$ **uniform** $0\ d']$

if $0 \leq s \leq d''$ **then do** $\{() \leftsquigarrow \wr () \mapsto 1/d'' \int$; $\square\}$ **else** $\wr \int$

$[$**let** $d' = d''$; **let** $s' = s]$

Automatic disintegrator in action

$$\text{perform}\ (\mathbf{do}\ \{d \leftsquigarrow \mathbf{uniform}\ 1\ 3;\ s \leftsquigarrow \mathbf{uniform}\ 0\ d;\ \mathbf{return}\ (s,d)\})^{[]}$$

$$[d' \leftsquigarrow \mathbf{uniform}\ 1\ 3]$$

$$\text{perform}\ (\mathbf{do}\ \{s \leftsquigarrow \mathbf{uniform}\ 0\ d';\ \mathbf{return}\ (s,d')\})$$

$$[d' \leftsquigarrow \mathbf{uniform}\ 1\ 3;\ s' \leftsquigarrow \mathbf{uniform}\ 0\ d']$$

$$\text{perform}\ (\mathbf{return}\ (s',d'))$$

$$\text{evaluate}\ (s',d') \Rightarrow (s',d')$$

$$\text{constrain-value}\ s'\ s$$

$$\text{constrain-outcome}\ (\mathbf{uniform}\ 0\ d')\ s$$

nondeterminism

$$\text{evaluate}\ 0 \Rightarrow 0$$

$$\text{evaluate}\ d'$$

$$\text{perform}\ (\mathbf{uniform}\ 1\ 3)$$

$$\Rightarrow d''$$

$$\mathbf{do}\ \{d'' \leftsquigarrow \mathbf{uniform}\ 1\ 3;\ \square\}$$

$$[\mathbf{let}\ d' = d'';\ s' \leftsquigarrow \mathbf{uniform}\ 0\ d']$$

$$\Rightarrow d''$$

$$\mathbf{if}\ 0 \le s \le d''\ \mathbf{then\ do}\ \{() \leftsquigarrow \wr () \mapsto {}^1\!/_{d''} \int;\ \square\}\ \mathbf{else}\ \wr \int$$

$$[\mathbf{let}\ d' = d'';\ \mathbf{let}\ s' = s]$$

Determinism requires inversion

$$\begin{aligned}&\textbf{do }\{d \leftsquigarrow \textbf{uniform } 0\ 1;\\ &\qquad s \leftsquigarrow \textbf{return } (2 \cdot d);\\ &\qquad \textbf{return } (s, d)\}\end{aligned}$$

Determinism requires inversion

$$\begin{aligned}&\textbf{do}\ \{d \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ &\qquad s \leftsquigarrow \textbf{return}\ (2 \cdot d);\\ &\qquad \textbf{return}\ (s, d)\}\end{aligned}$$

$$\begin{aligned}&\textbf{do}\ \{d_1 \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ &\qquad d_2 \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ &\qquad s \leftsquigarrow \textbf{return}\ (d_1 + d_2);\\ &\qquad \textbf{return}\ (s, (d_1, d_2))\}\end{aligned}$$

Determinism requires inversion

$$\begin{aligned}\textbf{do}\ \{&d \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ &s \leftsquigarrow \textbf{return}\ (2 \cdot d);\\ &\textbf{return}\ (s, d)\}\end{aligned}$$

$$\begin{aligned}\textbf{do}\ \{&d_1 \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ &d_2 \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ &s \leftsquigarrow \textbf{return}\ (d_1 + d_2);\\ &\textbf{return}\ (s, (d_1, d_2))\}\end{aligned}$$

$$\begin{aligned}\textbf{do}\ \{&d_1 \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ &d_2 \leftsquigarrow \lbag 1 \mapsto 1/2, 2 \mapsto 1/2 \rbag;\\ &s \leftsquigarrow \textbf{return}\ d_1^{d_2};\\ &\textbf{return}\ (s, (d_1, d_2))\}\end{aligned}$$

Summary

- Observe symptoms with hidden causes
- Infer probabilities by program transformations
- Specify disintegration by measure semantics
- Automate disintegration by lazy partial evaluation
- Future work: arrays (symbolically evaluated)
beyond **lebesgue**
prove correctness
computer algebra — please help!