Disintegrating probabilistic programs with arrays

Praveen Narayanan & Chung-chieh Shan

Indiana University, Bloomington

August 23, 2016

A tiny bit of context

Probabilistic programming automates conditioning.

A tiny bit of context

Probabilistic programming automates conditioning.

We can implement conditioning via disintegration.

A tiny bit of context

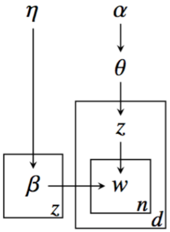
Probabilistic programming automates conditioning.

We can implement conditioning via disintegration.

Disintegration needs to handle programs with arrays.

Probabilistic inference

Model: Naive Bayes



$$
\begin{aligned}
\eta_w &= 1 && \text{for each } w \in \text{Words} \\
\beta_z &\leftsquigarrow \text{Dirichlet}(\eta) && \text{for each } z \in \text{Labels} \\
\alpha_z &= 1 && \text{for each } z \in \text{Labels} \\
\theta &\leftsquigarrow \text{Dirichlet}(\alpha) && \\
z_d &\leftsquigarrow \text{Categorical}(\theta) && \text{for each } d \in \text{Docs} \\
w_{dn} &\leftsquigarrow \text{Categorical}(\beta_{z_d}) && \text{for each } d \in \text{Docs}, \\
& && n \in \{1, \ldots, |d|\}
\end{aligned}
$$

Inference: Gibbs sampling

Deriving Gibbs for Naive Bayes

Resnik & Hardisty 2010

2.4.2 Choice of Priors and Simplifying the Joint Probability Expression

So why did we pick the Dirichlet distribution as our prior for $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_1$ and the Beta distribution as our prior for π? Let's look at what happens in the process of simplifying the joint distribution and see what happens to our estimates of $\boldsymbol{\theta}$ (where this can be either $\boldsymbol{\theta}_0$ or $\boldsymbol{\theta}_1$) and π once we observe some evidence (i.e. the words from a single document). Using (19) and (21) from above:

$$\begin{aligned} P(\pi|\mathbf{L};\gamma_{\pi 1},\gamma_{\pi 0}) &= P(\mathbf{L}|\pi)P(\pi|\gamma_{\pi 1},\gamma_{\pi 0}) && (28)\\ &\propto \left[\pi^{C_1}(1-\pi)^{C_0}\right]\left[\pi^{\gamma_{\pi 1}-1}(1-\pi)^{\gamma_{\pi 0}-1}\right] && (29)\\ &\propto \pi^{C_1+\gamma_{\pi 1}-1}(1-\pi)^{C_0+\gamma_{\pi 0}-1} && (30)\end{aligned}$$

Likewise, for $\boldsymbol{\theta}$ using (24) and (25) from above:

$$\begin{aligned} P(\boldsymbol{\theta}|\mathbf{W}_n;\boldsymbol{\gamma}_\theta) &= P(\mathbf{W}_n|\boldsymbol{\theta})P(\boldsymbol{\theta}|\boldsymbol{\gamma}_\theta)\\ &\propto \prod_{i=1}^{V}\theta_i^{\mathrm{W}_{ni}}\prod_{i=1}^{V}\theta_i^{\gamma_{\theta i}-1}\\ &\propto \prod_{i=1}^{V}\theta_i^{\mathrm{W}_{ni}+\gamma_{\theta i}-1} && (31)\end{aligned}$$

If we use the words in all of the documents of a given class, then we have:

$$P(\boldsymbol{\theta}_x|\mathbb{C}_x;\boldsymbol{\gamma}_\theta) \propto \prod_{i=1}^{V}\theta_{x,i}^{\mathcal{N}_{\mathbb{C}_x}(i)+\gamma_{\theta i}-1} \qquad (32)$$

$$P(\mathbb{C},\mathbf{L},\pi,\boldsymbol{\theta}_0,\boldsymbol{\theta}_1;\boldsymbol{\mu}) \propto \pi^{C_1+\gamma_{\pi 1}-1}(1-\pi)^{C_0+\gamma_{\pi 0}-1}\prod_{i=1}^{V}\theta_{0,i}^{\mathcal{N}_{\mathbb{C}_0}(i)+\gamma_{\theta i}-1}\theta_{1,i}^{\mathcal{N}_{\mathbb{C}_1}(i)+\gamma_{\theta i}-1} \qquad (33)$$

Integrate out hidden variable:

$$P(\mathbf{L},\mathbb{C},\boldsymbol{\theta}_0,\boldsymbol{\theta}_1;\boldsymbol{\mu}) \propto \frac{\Gamma(\gamma_{\pi 1}+\gamma_{\pi 0})}{\Gamma(\gamma_{\pi 1})\Gamma(\gamma_{\pi 0})}\frac{\Gamma(C_1+\gamma_{\pi 1})\Gamma(C_0+\gamma_{\pi 0})}{\Gamma(N+\gamma_{\pi 1}+\gamma_{\pi 0})}\prod_{i=1}^{V}\theta_{0,i}^{\mathcal{N}_{\mathbb{C}_0}(i)+\gamma_{\theta i}-1}\theta_{1,i}^{\mathcal{N}_{\mathbb{C}_1}(i)+\gamma_{\theta i}-1} \qquad (40)$$

The conditional distribution for our Gibbs sampler

$$\begin{aligned} P(\mathrm{L}_j|\mathbf{L}^{(-j)},\mathbb{C}^{(-j)},\boldsymbol{\theta}_0,\boldsymbol{\theta}_1;\boldsymbol{\mu}) &= \frac{P(\mathrm{L}_j,\mathbf{W}_j,\mathbf{L}^{(-j)},\mathbb{C}^{(-j)},\boldsymbol{\theta}_0,\boldsymbol{\theta}_1;\boldsymbol{\mu})}{P(\mathbf{L}^{(-j)},\mathbb{C}^{(-j)},\boldsymbol{\theta}_0,\boldsymbol{\theta}_1;\boldsymbol{\mu})} && (41)\\ &= \frac{P(\mathbf{L},\mathbb{C},\boldsymbol{\theta}_0,\boldsymbol{\theta}_1;\boldsymbol{\mu})}{P(\mathbf{L}^{(-j)},\mathbb{C}^{(-j)},\boldsymbol{\theta}_0,\boldsymbol{\theta}_1;\boldsymbol{\mu})} && (42)\end{aligned}$$

Becomes:

$$\Pr(\mathrm{L}_j=x|\mathbf{L}^{(-j)},\mathbb{C}^{(-j)},\boldsymbol{\theta}_0,\boldsymbol{\theta}_1;\boldsymbol{\mu}) = \frac{C_x+\gamma_{\pi x}-1}{N+\gamma_{\pi 1}+\gamma_{\pi 0}-1}\prod_{i=1}^{V}\theta_{x,i}^{\mathrm{W}_{ji}} \qquad (49)$$

And now we are ready to write the inference algorithm!

Gibbs sampler for Naive Bayes

2.5.4 Putting it all together

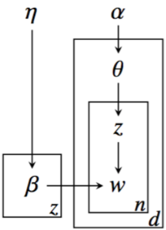
Initialization. Define the priors as in Section 2.2 and initialize them as described in Section 2.3.

for $t := 1$ to T **do**
    for $j := 1$ to N **do**
        if j is not a training document **then**
            Subtract j's word counts from the total word counts of whatever class it's currently a member of
            Subtract 1 from the count of documents with label L_j
            Assign a new label $L_j^{(t+1)}$ to document j as described at the end of Section 2.5.1
            Add 1 to the count of documents with label $L_j^{(t+1)}$
            Add j's word counts to the total word counts for class $L_j^{(t+1)}$
        end if
    end for
    $\mathbf{t}_0 :=$ vector of total word counts from class 0, including pseudocounts
    $\boldsymbol{\theta}_0 \sim \text{Dirichlet}(\mathbf{t}_0)$, as described in Section 2.5.2
    $\mathbf{t}_1 :=$ vector of total word counts from class 1, including pseudocounts
    $\boldsymbol{\theta}_1 \sim \text{Dirichlet}(\mathbf{t}_1)$, as described in Section 2.5.2
end for

Resnik & Hardisty 2010

Probabilistic inference

Model: LDA



$\eta_w = 1$		for each $w \in$ Words
$\beta_z \leftsquigarrow$ Dirichlet(η)		for each $z \in$ Labels
$\alpha_z = 1$		for each $z \in$ Labels
$\theta_d \leftsquigarrow$ Dirichlet(α)		for each $d \in$ Docs
$z_{dn} \leftsquigarrow$ Categorical(θ_d)		for each $d \in$ Docs, $n \in \{1, \ldots, \lvert d \rvert\}$
$w_{dn} \leftsquigarrow$ Categorical$(\beta_{z_{dn}})$		for each $d \in$ Docs, $n \in \{1, \ldots, \lvert d \rvert\}$

Inference: Gibbs sampling

Gibbs sampler for LDA

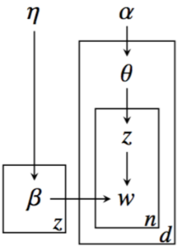
⊡ initialisation
zero all count variables, $n_m^{(k)}, n_m, n_k^{(t)}, n_k$
for all documents $m \in [1, M]$ **do**
  for all words $n \in [1, N_m]$ in document m **do**
    sample topic index $z_{m,n}{=}k \sim \text{Mult}(1/K)$
    increment document–topic count: $n_m^{(k)} + 1$
    increment document–topic sum: $n_m + 1$
    increment topic–term count: $n_k^{(t)} + 1$
    increment topic–term sum: $n_k + 1$
  end for
end for
⊡ Gibbs sampling over burn-in period and sampling period
while not finished **do**
  for all documents $m \in [1, M]$ **do**
    for all words $n \in [1, N_m]$ in document m **do**
      ⊡ for the current assignment of k to a term t for word $w_{m,n}$:
      decrement counts and sums: $n_m^{(k)} - 1; n_m - 1; n_k^{(t)} - 1; n_k - 1$
      ⊡ multinomial sampling acc. to Eq. 79 (decrements from previous step):
      sample topic index $\tilde{k} \sim p(z_i | \vec{z}_{\neg i}, \vec{w})$
      ⊡ use the new assignment of $z_{m,n}$ to the term t for word $w_{m,n}$ to:
      increment counts and sums: $n_m^{(\tilde{k})} + 1; n_m + 1; n_{\tilde{k}}^{(t)} + 1; n_{\tilde{k}} + 1$
    end for
  end for
  ⊡ check convergence and read out parameters
  if converged and L sampling iterations since last read out **then**
    ⊡ the different parameters read outs are averaged.
    read out parameter set $\underline{\Phi}$ according to Eq. 82
    read out parameter set $\underline{\Theta}$ according to Eq. 83
  end if
end while

Fig. 8. Gibbs sampling algorithm for latent Dirichlet allocation

Parameter estimation for text analysis, G. Heinrich (2005)

Probabilistic inference

Model: LDA



$$
\begin{aligned}
\eta_w &= 1 && \text{for each } w \in \text{Words} \\
\beta_z &\leftsquigarrow \text{Dirichlet}(\eta) && \text{for each } z \in \text{Labels} \\
\alpha_z &= 1 && \text{for each } z \in \text{Labels} \\
\theta_d &\leftsquigarrow \text{Dirichlet}(\alpha) && \text{for each } d \in \text{Docs} \\
z_{dn} &\leftsquigarrow \text{Categorical}(\theta_d) && \text{for each } d \in \text{Docs}, \\
&&& \qquad n \in \{1, \ldots, |d|\} \\
w_{dn} &\leftsquigarrow \text{Categorical}(\beta_{z_{dn}}) && \text{for each } d \in \text{Docs}, \\
&&& \qquad n \in \{1, \ldots, |d|\}
\end{aligned}
$$

Inference: Variational EM

Variational EM for LDA

initialize $\phi_{ni}^{0} := 1/k$ for all i and n
initialize $\gamma_i := \alpha_i + N/k$ for all i
repeat
  for $n = 1$ **to** N
    for $i = 1$ **to** k
      $\phi_{ni}^{t+1} := \beta_{iw_n} \exp(\Psi(\gamma_i^t))$
    normalize ϕ_n^{t+1} to sum to 1.
  $\gamma^{t+1} := \alpha + \sum_{n=1}^{N} \phi_n^{t+1}$
until convergence

Figure 6: A variational inference algorithm for LDA.

Blei et al (2003)

Probabilistic programming

Decouple model from inference

Probabilistic programming

Decouple model from inference

Perform inference via *conditioning*

Conditioning a "noisy measurement"

$$\begin{array}{l}\textbf{do } \{x \leftsquigarrow \textbf{normal } 2\ 3\textbf{;}\\ \quad n \leftsquigarrow \textbf{normal } 0\ 1\textbf{;}\\ \quad \textbf{return } (x + n, n)\}\end{array}$$

$\Pr(A, B) = \Pr(A \mid B) \Pr(B)$

$\Pr(A)$ is the distribution over the sum

$\Pr(B)$ is the distribution over the noise

Inference question: what is $\Pr(B \mid A)$?

Conditioning via disintegration

Known to statisticians
(Pachl 1978, Chang & Pollard 1997, Fremlin 2000)

Conditioning via disintegration

Known to statisticians
(Pachl 1978, Chang & Pollard 1997, Fremlin 2000)

Algorithm by Chung-chieh Shan and Norman Ramsey (2016)
`http://homes.soic.indiana.edu/ccshan/rational/disintegrator.pdf`

The input language

$$
\begin{array}{lrl}
\text{Variables} & x & \\
\text{Real numbers} & r \in \mathbb{R} & \\
\text{Terms} & e ::= & x \\
& \mid & r \mid \mathbf{-}e \mid e \mathbin{\mathbf{+}} e \mid \mathbf{normalD}\ r\ r\ e \\
& \mid & \mathbf{()} \mid \mathbf{(}e_1\mathbf{,}e_2\mathbf{)} \mid \mathbf{fst}\ e \mid \mathbf{snd}\ e \\
& \mid & \mathbf{lebesgue} \mid \mathbf{normal}\ r\ r \\
& \mid & \mathbf{return}\ e \mid \mathbf{do}\ \{x \leftsquigarrow e\mathbf{;}\ e\} \mid \mathbf{do}\ \{\mathbf{factor}\ e\mathbf{;}\ e\} \\
\text{Types} & \alpha, \beta ::= & \mathbb{1} \mid \mathbb{R} \mid \alpha \times \beta \mid \mathbb{M}\ \alpha
\end{array}
$$

Disintegration decomposes measures

$$\textbf{disintegrate} : \mathbb{M}\ (\alpha \times \beta) \to (\mathbb{M}\ \alpha\ ,\ \alpha \to \mathbb{M}\ \beta)$$

Disintegration decomposes measures

$$\mathbf{disintegrate} : \mathbb{M}\ (\alpha \times \beta) \to (\mathbb{M}\ \alpha\ ,\ \alpha \to \mathbb{M}\ \beta)$$

$$m \equiv \mathbf{do}\ \{$$

Disintegration decomposes measures

$$\textbf{disintegrate} : \mathbb{M}\ (\alpha \times \beta) \to (\mathbb{M}\ \alpha\ ,\ \alpha \to \mathbb{M}\ \beta)$$

$$m \equiv \textbf{do}\ \{t \leftsquigarrow \mathit{fst}\ (\textbf{disintegrate}\ m);$$

Disintegration decomposes measures

$$\textbf{disintegrate} : \mathbb{M}\ (\alpha \times \beta) \to (\mathbb{M}\ \alpha\ ,\ \alpha \to \mathbb{M}\ \beta)$$

$$\begin{aligned} m \equiv \textbf{do}\ \{ & t \leftsquigarrow \mathit{fst}\ (\textbf{disintegrate}\ m)\textbf{;} \\ & p \leftsquigarrow \mathit{snd}\ (\textbf{disintegrate}\ m)\ t\textbf{;} \end{aligned}$$

Disintegration decomposes measures

$$\mathbf{disintegrate} : \mathbb{M}\ (\alpha \times \beta) \to (\mathbb{M}\ \alpha\ ,\ \alpha \to \mathbb{M}\ \beta)$$

$$\begin{aligned} m \equiv \mathbf{do}\ \{ & t \leftsquigarrow \mathit{fst}\ (\mathbf{disintegrate}\ m)\mathbf{;} \\ & p \leftsquigarrow \mathit{snd}\ (\mathbf{disintegrate}\ m)\ t\mathbf{;} \\ & \mathbf{return}\ (t, p)\} \end{aligned}$$

Disintegration decomposes measures

$$\textbf{disintegrate} : \mathbb{M}\ (\mathbb{R} \times \beta) \to (\mathbb{M}\ \mathbb{R}\ ,\ \mathbb{R} \to \mathbb{M}\ \beta)$$

$$\begin{aligned} m \equiv \textbf{do}\ \{&t \leftsquigarrow \textbf{lebesgue};\\ &p \leftsquigarrow \mathit{snd}\ (\textbf{disintegrate}\ m)\ t;\\ &\textbf{return}\ (t, p)\} \end{aligned}$$

Let's consider the special case

$$\textbf{disintegrate} : \mathbb{M}\ (\mathbb{R} \times \beta) \to (\mathbb{R} \to \mathbb{M}\ \beta)$$

$$\begin{aligned} m \equiv \textbf{do}\ \{ & t \leftsquigarrow \textbf{lebesgue;} \\ & p \leftsquigarrow \textbf{disintegrate}\ m\ t\textbf{;} \\ & \textbf{return}\ (t, p) \} \end{aligned}$$

Let's consider the special case

$$\textbf{disintegrate} : \mathbb{M}\ (\mathbb{R} \times \beta) \to (\mathbb{R} \to \mathbb{M}\ \beta)$$

$$\begin{aligned} m \equiv \textbf{do}\ \{ & t \leftsquigarrow \textbf{lebesgue;} \\ & p \leftsquigarrow \textbf{disintegrate}\ m\ t\textbf{;} \\ & \textbf{return}\ (t, p) \} \end{aligned}$$

This is our *specification*.

Conditioning a noisy measurement

$$
\begin{aligned}
&m = \\
&\mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3\mathbf{;} \\
&\qquad n \leftsquigarrow \mathbf{normal}\ 0\ 1\mathbf{;} \\
&\qquad \mathbf{return}\ (x \mathbin{\mathbf{+}} n\,\mathbf{,}\,n)\}
\end{aligned}
$$

Conditioning a noisy measurement

$m =$
$\textbf{do}\ \{x \leftsquigarrow \textbf{normal}\ 2\ 3\textbf{;}$
$\quad n \leftsquigarrow \textbf{normal}\ 0\ 1\textbf{;}$
$\quad \textbf{return}\ (x + n, n)\}$

$\textbf{disintegrate}\ m\ t = \textbf{do}\ \{\ell \leftsquigarrow \textbf{normal}\ 0\ 1\textbf{;}$
$\quad \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (t - \ell))\textbf{;}$
$\quad x \leftsquigarrow \textbf{return}\ (t - \ell)\textbf{;}$
$\quad n \leftsquigarrow \textbf{return}\ \ell\textbf{;}$
$\quad \textbf{return}\ n\}$

Conditioning a noisy measurement

$$m = \mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3;\ n \leftsquigarrow \mathbf{normal}\ 0\ 1;\ \mathbf{return}\ (x + n, n)\}$$

$$\mathbf{disintegrate}\ m\ t = \mathbf{do}\ \{\ell \leftsquigarrow \mathbf{normal}\ 0\ 1;\ \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (t - \ell));\ x \leftsquigarrow \mathbf{return}\ (t - \ell);\ n \leftsquigarrow \mathbf{return}\ \ell;\ \mathbf{return}\ n\}$$

From the specification:

$$\mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3;\ n \leftsquigarrow \mathbf{normal}\ 0\ 1;\ \mathbf{return}\ (x + n, n)\} \equiv \mathbf{do}\ \{t \leftsquigarrow \mathbf{lebesgue};\ n \leftsquigarrow \mathbf{do}\ \{\ell \leftsquigarrow \mathbf{normal}\ 0\ 1;\ \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (t - \ell));\ x \leftsquigarrow \mathbf{return}\ (t - \ell);\ n \leftsquigarrow \mathbf{return}\ \ell;\ \mathbf{return}\ n\};\ \mathbf{return}\ (t, n)\}$$

The extended language used by the disintegrator

Variables	x
Locations	$\overline{x}$
Real numbers $r \in \mathbb{R}$	
Bindings	$b ::= \overline{x} \leftsquigarrow e \mid \mathbf{factor}\ e$
Heap	$h ::= [b_1; b_2; \ldots; b_n]$
Terms	$e ::= x \mid \overline{x}$
	$\mid\ r \mid -e \mid e + e \mid \mathbf{normalD}\ r\ r\ e$
	$\mid\ () \mid (e_1, e_2) \mid \mathbf{fst}\ e \mid \mathbf{snd}\ e$
	$\mid\ \mathbf{lebesgue} \mid \mathbf{normal}\ r\ r$
	$\mid\ \mathbf{return}\ e \mid \mathbf{do}\ \{x \leftsquigarrow e; e\} \mid \mathbf{do}\ \{\mathbf{factor}\ e; e\}$
Types	$\alpha, \beta ::= \mathbb{1} \mid \mathbb{R} \mid \alpha \times \beta \mid \mathbb{M}\ \alpha$

The disintegrator is a lazy partial evaluator

$\ggg$ ("perform") : $\mathit{heap} \to \lceil \mathbb{M}\,\alpha \rceil \to (\mathit{emission} \times \mathit{heap} \times \lfloor \alpha \rfloor)$

$\rhd$ ("evaluate") : $\mathit{heap} \to \lceil \alpha \rceil \to (\mathit{emission} \times \mathit{heap} \times \lfloor \alpha \rfloor)$

$\lll$ ("constrain outcome") : $\mathit{heap} \to \lceil \mathbb{M}\,\alpha \rceil \to \lfloor \alpha \rfloor \to (\mathit{emission} \times \mathit{heap})$

$\lhd$ ("constrain value") : $\mathit{heap} \to \lceil \alpha \rceil \to \lfloor \alpha \rfloor \to (\mathit{emission} \times \mathit{heap})$

The definition of **disintegrate**

$$
\begin{array}{lll}
\gg\!\!\rhd \text{ (``perform'')} & : \mathit{heap} \to \lceil \mathbb{M}\ \alpha \rceil \to & (\mathit{emission} \times \mathit{heap} \times \lfloor \alpha \rfloor) \\
\rhd \text{ (``evaluate'')} & : \mathit{heap} \to \lceil \alpha \rceil \quad \to & (\mathit{emission} \times \mathit{heap} \times \lfloor \alpha \rfloor) \\
\lll\!\!\lhd \text{ (``constrain outcome'')} & : \mathit{heap} \to \lceil \mathbb{M}\ \alpha \rceil \to \lfloor \alpha \rfloor \to & (\mathit{emission} \times \mathit{heap}) \\
\lhd \text{ (``constrain value'')} & : \mathit{heap} \to \lceil \alpha \rceil \quad \to \lfloor \alpha \rfloor \to & (\mathit{emission} \times \mathit{heap})
\end{array}
$$

$$
\begin{aligned}
&\mathbf{disintegrate} : \mathbb{M}\ (\mathbb{R} \times \beta) \to (\mathbb{R} \to \mathbb{M}\ \beta) \\
&\mathbf{disintegrate}\ m\ t = \mathit{let}\ \big(e_1\ , h_1\ , v\big) = \gg\!\!\rhd\ []\ \ m \\
&\qquad\qquad\qquad\qquad\quad\ \ \big(e_2\ , h_2\big) = \lhd\ h_1\ (\mathit{fst}\ v)\ t \\
&\qquad\qquad\qquad\quad\ \ \mathit{in}\ \mathbf{do}\ \{e_1; e_2; h_2; \mathbf{return}\ (\mathit{snd}\ v)\}
\end{aligned}
$$

The definition of **disintegrate**

$$
\begin{array}{llllll}
\ggg & (\text{“perform”}) & : \mathit{heap} \to \lceil \mathbb{M}\ \alpha \rceil \to & & (\mathit{emission} \times \mathit{heap} \times \lfloor \alpha \rfloor) \\
\rhd & (\text{“evaluate”}) & : \mathit{heap} \to \lceil \alpha \rceil \quad\ \to & & (\mathit{emission} \times \mathit{heap} \times \lfloor \alpha \rfloor) \\
\lll & (\text{“constrain outcome”}) & : \mathit{heap} \to \lceil \mathbb{M}\ \alpha \rceil \to & \lfloor \alpha \rfloor \to & (\mathit{emission} \times \mathit{heap}) \\
\lhd & (\text{“constrain value”}) & : \mathit{heap} \to \lceil \alpha \rceil \quad\ \to & \lfloor \alpha \rfloor \to & (\mathit{emission} \times \mathit{heap})
\end{array}
$$

$$\textbf{disintegrate} : \mathbb{M}\ (\mathbb{R} \times \beta) \to (\mathbb{R} \to \mathbb{M}\ \beta)$$

$$
\begin{aligned}
\textbf{disintegrate}\ m\ t = \mathit{let}\ & \big(e_1\ , h_1\ , v\big) = \ggg\ [\,]\ \ m \\
& \big(e_2\ , h_2\big) = \lhd\ h_1\ (\mathit{fst}\ v)\ t \\
\mathit{in}\ & \textbf{do}\ \{e_1; e_2; h_2; \textbf{return}\ (\mathit{snd}\ v)\}
\end{aligned}
$$

For example, consider the “noisy measurement” model

$$
\begin{aligned}
m = \textbf{do}\ \{& x \leftsquigarrow \textbf{normal}\ 2\ 3; \\
& n \leftsquigarrow \textbf{normal}\ 0\ 1; \\
& \textbf{return}\ (\,x + n\,,n\,)\}
\end{aligned}
$$

Step 1: $\gg$ [] m

Store probabilistic choices and weights on the heap

$$\left[\vphantom{\begin{array}{c}a\\a\\a\\a\\a\\a\\a\end{array}} \right. \Longrightarrow$$

Step 1: $\gg\!\!\!\triangleright\ []\ m$

Store probabilistic choices and weights on the heap

$$\left[\begin{array}{l} \mathrel{\triangleright\!\!\!>}\ [] \\ \\ \\ \\ \\ \Longrightarrow \end{array}\right.$$

$$[\![\mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3\mathbf{;}\\ \quad n \leftsquigarrow \mathbf{normal}\ 0\ 1\mathbf{;}\\ \quad \mathbf{return}\ (x + n, n)\}]\!]$$

Step 1: $\gg [\,]\ m$

Store probabilistic choices and weights on the heap

$$
\left[\begin{array}{ll}
\gg [\,] & [\![\mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
& \quad n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
& \quad \mathbf{return}\ (x + n, n)\}]\!]\\
\gg [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3] & [\![\mathbf{do}\ \{n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
& \quad \mathbf{return}\ (\overline{x} + n, n)\}]\!]\\
\\
\Longrightarrow &
\end{array}\right.
$$

Step 1: $\rhd\!\!\rhd \; [\,] \; m$

Store probabilistic choices and weights on the heap

$$
\left[\begin{array}{ll}
\rhd\!\!\rhd\ [\,] & [\![\mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
& \qquad n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
& \qquad \mathbf{return}\ (x + n, n)\}]\!]\\
\rhd\!\!\rhd\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3] & [\![\mathbf{do}\ \{n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
& \qquad \mathbf{return}\ (\overline{x} + n, n)\}]\!]\\
\rhd\!\!\rhd\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3; \overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1] & [\![\mathbf{return}\ (\overline{x} + \overline{n}, \overline{n})]\!]\\
\\
\Longrightarrow &
\end{array}\right.
$$

Step 1: $\gg [\,]\ m$

Store probabilistic choices and weights on the heap

$$
\left[\begin{array}{ll}
\gg [\,] & [\![\mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3; \\
 & \quad n \leftsquigarrow \mathbf{normal}\ 0\ 1; \\
 & \quad \mathbf{return}\ (x + n, n)\}]\!] \\
\gg [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3] & [\![\mathbf{do}\ \{n \leftsquigarrow \mathbf{normal}\ 0\ 1; \\
 & \quad \mathbf{return}\ (\overline{x} + n, n)\}]\!] \\
\gg [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3; \overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1] & [\![\mathbf{return}\ (\overline{x} + \overline{n}, \overline{n})]\!] \\
\rhd\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3; \overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1] & [\![(\overline{x} + \overline{n}, \overline{n})]\!] \\
\Longrightarrow\ ([\,], [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3; \overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1], [\![(\overline{x} + \overline{n}, \overline{n})]\!]) &
\end{array}\right.
$$

Step 2: $\triangleleft\ h_1\ (fst\ v)\ t$

$\triangleright$ lazily emits bindings

$\triangleleft$ clamps bindings and pushes weights onto the heap

Step 2: $\triangleleft\ h_1\ (fst\ v)\ t$

$\triangleleft\ [\overline{x} \leftsquigarrow \textbf{normal}\ 2\ 3\ ; \overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1]\ [\![\overline{x} + \overline{n}]\!]\ t$

$\Longrightarrow$

$\Longrightarrow$

$\Longrightarrow$

$\Longrightarrow$

$\Longrightarrow$

$\triangleright$ lazily emits bindings

$\triangleleft$ clamps bindings and pushes weights onto the heap

Step 2: $\lhd\; h_1\; (fst\; v)\; t$

$\lhd\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3\ ; \overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1]\ [\![\overline{x} + \overline{n}]\!]\ t$

$\rhd\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3\ ; \overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1]\ [\![\overline{n}]\!]$

$\gg\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3]\ [\![\mathbf{normal}\ 0\ 1]\!]$

$\Longrightarrow\ ([\ell \leftsquigarrow \mathbf{normal}\ 0\ 1]\ ,\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3]\ ,\ [\![\ell]\!])$

$\Longrightarrow$

$\Longrightarrow$

$\Longrightarrow$

$\Longrightarrow$

$\rhd$ lazily emits bindings

$\lhd$ clamps bindings and pushes weights onto the heap

Step 2: $\lhd\ h_1\ (fst\ v)\ t$

$\lhd\ [\overline{x} \leftsquigarrow \textbf{normal}\ 2\ 3\ ; \overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1]\ [\![\overline{x} + \overline{n}]\!]\ t$

$\rhd\ [\overline{x} \leftsquigarrow \textbf{normal}\ 2\ 3\ ; \overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1]\ [\![\overline{n}]\!]$

$\rhd\!\!\!\rhd\ [\overline{x} \leftsquigarrow \textbf{normal}\ 2\ 3]\ [\![\textbf{normal}\ 0\ 1]\!]$

$\Longrightarrow ([\ell \leftsquigarrow \textbf{normal}\ 0\ 1]\ ,\ [\overline{x} \leftsquigarrow \textbf{normal}\ 2\ 3]\ ,\ [\![\ell]\!])$

$\Longrightarrow ([\ell \leftsquigarrow \textbf{normal}\ 0\ 1]\ ,\ [\overline{x} \leftsquigarrow \textbf{normal}\ 2\ 3\ ; \overline{n} \leftsquigarrow \textbf{return}\ [\![\ell]\!]]\ ,\ [\![\ell]\!])$

$\Longrightarrow$

$\Longrightarrow$

$\Longrightarrow$

$\rhd$ lazily emits bindings

$\lhd$ clamps bindings and pushes weights onto the heap

Step 2: $\lhd\ h_1\ (\mathit{fst}\ v)\ t$

$$\lhd\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3\ ; \overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1]\ [\![\overline{x} + \overline{n}]\!]\ t$$

$$\rhd\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3\ ; \overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1]\ [\![\overline{n}]\!]$$

$$\unrhd\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3]\ [\![\mathbf{normal}\ 0\ 1]\!]$$

$$\Longrightarrow\ ([\ell \leftsquigarrow \mathbf{normal}\ 0\ 1]\ ,\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3]\ ,\ [\![\ell]\!])$$

$$\Longrightarrow\ ([\ell \leftsquigarrow \mathbf{normal}\ 0\ 1]\ ,\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3\ ; \overline{n} \leftsquigarrow \mathbf{return}\ [\![\ell]\!]]\ ,\ [\![\ell]\!])$$

$$\lhd\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3\ ; \overline{n} \leftsquigarrow \mathbf{return}\ [\![\ell]\!]]\ [\![\overline{x}]\!]\ (t - [\![\ell]\!])$$

$$\lll\ []\ [\![\mathbf{normal}\ 2\ 3]\!]\ (t - [\![\ell]\!])$$

$$\Longrightarrow\ ([]\ ,\ [\mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (t - [\![\ell]\!]))])$$

$$\Longrightarrow$$

$$\Longrightarrow$$

$\rhd$ lazily emits bindings

$\lhd$ clamps bindings and pushes weights onto the heap

Step 2: $\lhd\ h_1\ (\mathit{fst}\ v)\ t$

$$\lhd\ [\overline{x} \leftsquigarrow \textbf{normal}\ 2\ 3\,;\overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1]\ [\![\overline{x} + \overline{n}]\!]\ t$$

$$\rhd\ [\overline{x} \leftsquigarrow \textbf{normal}\ 2\ 3\,;\overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1]\ [\![\overline{n}]\!]$$

$$\unrhd\ [\overline{x} \leftsquigarrow \textbf{normal}\ 2\ 3]\ [\![\textbf{normal}\ 0\ 1]\!]$$

$$\Longrightarrow\ ([\ell \leftsquigarrow \textbf{normal}\ 0\ 1]\ ,\ [\overline{x} \leftsquigarrow \textbf{normal}\ 2\ 3]\ ,\ [\![\ell]\!])$$

$$\Longrightarrow\ ([\ell \leftsquigarrow \textbf{normal}\ 0\ 1]\ ,\ [\overline{x} \leftsquigarrow \textbf{normal}\ 2\ 3\,;\overline{n} \leftsquigarrow \textbf{return}\ [\![\ell]\!]]\ ,\ [\![\ell]\!])$$

$$\lhd\ [\overline{x} \leftsquigarrow \textbf{normal}\ 2\ 3\,;\overline{n} \leftsquigarrow \textbf{return}\ [\![\ell]\!]]\ [\![\overline{x}]\!]\ (t - [\![\ell]\!])$$

$$\lll\ []\ [\![\textbf{normal}\ 2\ 3]\!]\ (t - [\![\ell]\!])$$

$$\Longrightarrow\ ([]\ ,\ [\textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (t - [\![\ell]\!]))])$$

$$\Longrightarrow\ \left([]\ ,\ \begin{bmatrix}\textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (t - [\![\ell]\!]));\\ \overline{x} \leftsquigarrow \textbf{return}\ (t - [\![\ell]\!]);\\ \overline{n} \leftsquigarrow \textbf{return}\ [\![\ell]\!]\end{bmatrix}\right)$$

$$\Longrightarrow$$

$\rhd$ lazily emits bindings

$\lhd$ clamps bindings and pushes weights onto the heap

Step 2: $\lhd\ h_1\ (fst\ v)\ t$

$$\lhd\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3\ ;\overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1]\ \llbracket \overline{x} + \overline{n} \rrbracket\ t$$

$$\rhd\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3\ ;\overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1]\ \llbracket \overline{n} \rrbracket$$

$$\gg\!\rhd\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3]\ \llbracket \mathbf{normal}\ 0\ 1 \rrbracket$$

$$\Longrightarrow\ ([\ell \leftsquigarrow \mathbf{normal}\ 0\ 1]\ ,\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3]\ ,\ \llbracket \ell \rrbracket)$$

$$\Longrightarrow\ ([\ell \leftsquigarrow \mathbf{normal}\ 0\ 1]\ ,\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3\ ;\overline{n} \leftsquigarrow \mathbf{return}\ \llbracket \ell \rrbracket]\ ,\ \llbracket \ell \rrbracket)$$

$$\lhd\ [\overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3\ ;\overline{n} \leftsquigarrow \mathbf{return}\ \llbracket \ell \rrbracket]\ \llbracket \overline{x} \rrbracket\ (t - \llbracket \ell \rrbracket)$$

$$\lll\ []\ \llbracket \mathbf{normal}\ 2\ 3 \rrbracket\ (t - \llbracket \ell \rrbracket)$$

$$\Longrightarrow\ ([]\ ,\ [\mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (t - \llbracket \ell \rrbracket))])$$

$$\Longrightarrow\ \left([]\ ,\ \begin{bmatrix} \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (t - \llbracket \ell \rrbracket)); \\ \overline{x} \leftsquigarrow \mathbf{return}\ (t - \llbracket \ell \rrbracket); \\ \overline{n} \leftsquigarrow \mathbf{return}\ \llbracket \ell \rrbracket \end{bmatrix}\right)$$

$$\Longrightarrow\ \left([\ell \leftsquigarrow \mathbf{normal}\ 0\ 1]\ ,\ \begin{bmatrix} \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (t - \llbracket \ell \rrbracket)); \\ \overline{x} \leftsquigarrow \mathbf{return}\ (t - \llbracket \ell \rrbracket); \\ \overline{n} \leftsquigarrow \mathbf{return}\ \llbracket \ell \rrbracket \end{bmatrix}\right)$$

$\rhd$ lazily emits bindings

$\lhd$ clamps bindings and pushes weights onto the heap

Step 3: putting it together

$$
\begin{aligned}
m = \mathbf{do}\ \{ & x \leftsquigarrow \mathbf{normal}\ 2\ 3; \\
& n \leftsquigarrow \mathbf{normal}\ 0\ 1; \\
& \mathbf{return}\ (x + n, n) \}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{disintegrate}\ m\ t = \mathit{let}\ & (e_1, h_1, v) = \rhd\!\!\!\rhd\ []\ m \\
& (e_2, h_2) = \lhd\ h_1\ (\mathit{fst}\ v)\ t \\
\mathit{in}\ & \mathbf{do}\ \{e_1; e_2; h_2; \mathbf{return}\ (\mathit{snd}\ v)\}
\end{aligned}
$$

$$
\begin{aligned}
= \mathbf{do}\ \{ & \ell \leftsquigarrow \mathbf{normal}\ 0\ 1; \\
& \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (t - \ell)); \\
& x \leftsquigarrow \mathbf{return}\ (t - \ell); \\
& n \leftsquigarrow \mathbf{return}\ \ell; \\
& \mathbf{return}\ n \}
\end{aligned}
$$

A pair of noisy measurements

1-D case

2-D case

$$
\begin{aligned}
m = \textbf{do}\ \{&x \leftsquigarrow \textbf{normal}\ 2\ 3\textbf{;}\\
&n \leftsquigarrow \textbf{normal}\ 0\ 1\textbf{;}\\
&\textbf{return}\ (\,x + n\,,\,n\,)\}
\end{aligned}
$$

A pair of noisy measurements

1-D case

$$\begin{aligned} m = \mathbf{do}\ \{ & x \leftarrow\!\sim \mathbf{normal}\ 2\ 3; \\ & n \leftarrow\!\sim \mathbf{normal}\ 0\ 1; \\ & \mathbf{return}\ (x + n, n)\} \end{aligned}$$

2-D case

$$\begin{aligned} m = \mathbf{do}\ \{ & n \leftarrow\!\sim \mathbf{normal}\ 0\ 1; \\ & p \leftarrow\!\sim \mathbf{do}\ \{ x_1 \leftarrow\!\sim \mathbf{normal}\ 2\ 3; \\ & \qquad\qquad x_2 \leftarrow\!\sim \mathbf{normal}\ 2\ 3; \\ & \qquad\qquad \mathbf{return}\ (x_1 + n, x_2 + n)\}; \\ & \mathbf{return}\ (p, n)\} \end{aligned}$$

A pair of noisy measurements

1-D case

$$m = \mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3;\ n \leftsquigarrow \mathbf{normal}\ 0\ 1;\ \mathbf{return}\ (x+n, n)\}$$

$$\begin{aligned}&\mathbf{disintegrate}\ m\ t = \\ &\mathbf{do}\ \{\ell \leftsquigarrow \mathbf{normal}\ 0\ 1; \\ &\quad \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (t-\ell)); \\ &\quad x \leftsquigarrow \mathbf{return}\ (t-\ell); \\ &\quad n \leftsquigarrow \mathbf{return}\ \ell; \\ &\quad \mathbf{return}\ n\}\end{aligned}$$

2-D case

$$\begin{aligned}m = \mathbf{do}\ \{&n \leftsquigarrow \mathbf{normal}\ 0\ 1; \\ &p \leftsquigarrow \mathbf{do}\ \{x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3; \\ &\qquad\quad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3; \\ &\qquad\quad \mathbf{return}\ (x_1+n, x_2+n)\}; \\ &\mathbf{return}\ (p, n)\}\end{aligned}$$

A pair of noisy measurements

1-D case

$$
\begin{aligned}
&m = \textbf{do}\ \{x \leftsquigarrow \textbf{normal}\ 2\ 3;\\
&\qquad\quad n \leftsquigarrow \textbf{normal}\ 0\ 1;\\
&\qquad\quad \textbf{return}\ (x+n, n)\}
\end{aligned}
$$

$$
\begin{aligned}
&\textbf{disintegrate}\ m\ t =\\
&\textbf{do}\ \{\ell \leftsquigarrow \textbf{normal}\ 0\ 1;\\
&\quad \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (t-\ell));\\
&\quad x \leftsquigarrow \textbf{return}\ (t-\ell);\\
&\quad n \leftsquigarrow \textbf{return}\ \ell;\\
&\quad \textbf{return}\ n\}
\end{aligned}
$$

2-D case

$$
\begin{aligned}
&m = \textbf{do}\ \{n \leftsquigarrow \textbf{normal}\ 0\ 1;\\
&\qquad\quad p \leftsquigarrow \textbf{do}\ \{x_1 \leftsquigarrow \textbf{normal}\ 2\ 3;\\
&\qquad\qquad\qquad\quad x_2 \leftsquigarrow \textbf{normal}\ 2\ 3;\\
&\qquad\qquad\qquad\quad \textbf{return}\ (x_1+n, x_2+n)\};\\
&\qquad\quad \textbf{return}\ (p, n)\}
\end{aligned}
$$

$$
\begin{aligned}
&\textbf{disintegrate}\ m\ t =\\
&\textbf{do}\ \{\ell \leftsquigarrow \textbf{normal}\ 0\ 1;\\
&\quad n \leftsquigarrow \textbf{return}\ [\![\ell]\!];\\
&\quad \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{fst}\ t - [\![\ell]\!]));\\
&\quad x_1 \leftsquigarrow \textbf{return}\ (\textbf{fst}\ t - [\![\ell]\!]);\\
&\quad \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{snd}\ t - [\![\ell]\!]));\\
&\quad x_2 \leftsquigarrow \textbf{return}\ (\textbf{snd}\ t - [\![\ell]\!]);\\
&\quad p \leftsquigarrow \textbf{return}\ t;\\
&\quad \textbf{return}\ n\}
\end{aligned}
$$

Step 1: $\rhd\!\!\!\rhd\ []\ m$

1-D case

2-D case

$$\left[\begin{array}{l} \rhd\!\!\!\rhd\ []\ [\![\mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3; \\ \qquad\qquad n \leftsquigarrow \mathbf{normal}\ 0\ 1; \\ \qquad\qquad \mathbf{return}\ (x + n, n)\}]\!] \\ \quad \vdots \\ \Longrightarrow \left([],\ \begin{bmatrix} \overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3; \\ \overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1 \end{bmatrix},\ [\![(\overline{x} + \overline{n}, \overline{n})]\!] \right) \end{array}\right.$$

$$\left[\begin{array}{l} \\ \\ \\ \Longrightarrow \\ \\ \end{array}\right.$$

Step 1: $\rhd\!\!\!\rhd\ []\ m$

1-D case

$$
\left[
\begin{array}{l}
\rhd\!\!\!\rhd\ []\ [\![\mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
\qquad\qquad \mathbf{return}\ (x + n, n)\}]\!]\\
\quad\vdots\\
\Longrightarrow \left([],\ \begin{bmatrix} \overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3; \\ \overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1 \end{bmatrix},\ [\![(\overline{x} + \overline{n}, \overline{n})]\!] \right)
\end{array}
\right.
$$

2-D case

$$
\left[
\begin{array}{l}
\rhd\!\!\!\rhd\ []\ [\![\mathbf{do}\ \{n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
\qquad\qquad p \leftsquigarrow \mathbf{do}\ \{x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad\qquad\qquad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad\qquad\qquad \mathbf{return}\ (x_1 + n, x_2 + n)\};\\
\qquad\qquad \mathbf{return}\ (p, n)\}]\!]\\
\quad\vdots\\
\\
\Longrightarrow\\
\\
\end{array}
\right.
$$

Step 1: $\rhd\!\!\!\rhd \; [] \; m$

1-D case

$$
\left\lfloor\begin{array}{l}
\rhd\!\!\!\rhd \; [] \; [\![\mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
\qquad\qquad \mathbf{return}\ (x + n, n)\}]\!]\\
\quad\vdots\\
\Longrightarrow \left([], \begin{bmatrix} \overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3;\\ \overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1 \end{bmatrix}, [\![(\overline{x} + \overline{n}, \overline{n})]\!] \right)
\end{array}\right.
$$

2-D case

$$
\left\lfloor\begin{array}{l}
\rhd\!\!\!\rhd \; [] \; [\![\mathbf{do}\ \{n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
\qquad\qquad p \leftsquigarrow \mathbf{do}\ \{x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad\qquad\quad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad\qquad\quad \mathbf{return}\ (x_1 + n, x_2 + n)\};\\
\qquad\qquad \mathbf{return}\ (p, n)\}]\!]\\
\quad\vdots\\
\Longrightarrow \left([], \begin{bmatrix} \overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1;\\ \overline{p} \leftsquigarrow \mathbf{do}\ \{x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\ \qquad\quad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\ \qquad\quad \mathbf{return}\ (x_1 + \overline{n}, x_2 + \overline{n})\} \end{bmatrix}, [\![(\overline{p}, \overline{n})]\!] \right)
\end{array}\right.
$$

Step 2: $\lhd\ h_1\ (\mathit{fst}\ v)\ t$

1-D case

2-D case

$$
\left[\begin{array}{l}
\lhd \begin{bmatrix} \overline{x} \leftsquigarrow \textbf{normal}\ 2\ 3; \\ \overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1 \end{bmatrix} [\![\overline{x} + \overline{n}]\!]\ t \\
\left[\begin{array}{l} \rhd \begin{bmatrix} \overline{x} \leftsquigarrow \textbf{normal}\ 2\ 3; \\ \overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1 \end{bmatrix} [\![\overline{n}]\!] \\ \qquad\qquad \vdots \end{array}\right. \\
\left[\begin{array}{l} \lhd \begin{bmatrix} \overline{x} \leftsquigarrow \textbf{normal}\ 2\ 3; \\ \overline{n} \leftsquigarrow \textbf{return}\ [\![\ell]\!] \end{bmatrix} [\![\overline{x}]\!]\ (t - [\![\ell]\!]) \\ \qquad\qquad \vdots \end{array}\right. \\
\Longrightarrow \left([\ell \leftsquigarrow \textbf{normal}\ 0\ 1]\ ,\ \begin{bmatrix} \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (t - [\![\ell]\!])); \\ \overline{x} \leftsquigarrow \textbf{return}\ (t - [\![\ell]\!]); \\ \overline{n} \leftsquigarrow \textbf{return}\ [\![\ell]\!] \end{bmatrix} \right)
\end{array}\right.
$$

Step 2: $\lhd\ h_1\ (\mathit{fst}\ v)\ t$

1-D case

$$\lhd \begin{bmatrix} \overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3; \\ \overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1 \end{bmatrix} \llbracket \overline{x} + \overline{n} \rrbracket\ t$$

$$\rhd \begin{bmatrix} \overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3; \\ \overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1 \end{bmatrix} \llbracket \overline{n} \rrbracket$$

$$\vdots$$

$$\lhd \begin{bmatrix} \overline{x} \leftsquigarrow \mathbf{normal}\ 2\ 3; \\ \overline{n} \leftsquigarrow \mathbf{return}\ \llbracket \ell \rrbracket \end{bmatrix} \llbracket \overline{x} \rrbracket\ (t - \llbracket \ell \rrbracket)$$

$$\vdots$$

$$\Longrightarrow \left(\left[\ell \leftsquigarrow \mathbf{normal}\ 0\ 1 \right],\ \begin{bmatrix} \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (t - \llbracket \ell \rrbracket)); \\ \overline{x} \leftsquigarrow \mathbf{return}\ (t - \llbracket \ell \rrbracket); \\ \overline{n} \leftsquigarrow \mathbf{return}\ \llbracket \ell \rrbracket \end{bmatrix} \right)$$

2-D case

$$\lhd \begin{bmatrix} \overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1; \\ \overline{p} \leftsquigarrow \mathbf{do}\ \{x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3; \\ \qquad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3; \\ \qquad \mathbf{return}\ (x_1 + \overline{n}, x_2 + \overline{n})\} \end{bmatrix} \llbracket \overline{p} \rrbracket\ t$$

$$\vdots$$

$$\lhd \left[\overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1; \overline{x_1} \leftsquigarrow \mathbf{normal}\ 2\ 3; \overline{x_2} \leftsquigarrow \mathbf{normal}\ 2\ 3 \right] \llbracket (\overline{x_1} + \overline{n}, \overline{x_2} + \overline{n}) \rrbracket\ t$$

$$\lhd \left[\overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1; \overline{x_1} \leftsquigarrow \mathbf{normal}\ 2\ 3; \overline{x_2} \leftsquigarrow \mathbf{normal}\ 2\ 3 \right] \llbracket \overline{x_1} + \overline{n} \rrbracket\ (\mathbf{fst}\ t)$$

$$\vdots$$

$$\lhd \begin{bmatrix} \overline{n} \leftsquigarrow \mathbf{return}\ \llbracket \ell \rrbracket; \\ \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (\mathbf{fst}\ t - \llbracket \ell \rrbracket)); \\ \overline{x_1} \leftsquigarrow \mathbf{return}\ (\mathbf{fst}\ t - \llbracket \ell \rrbracket); \\ \overline{x_2} \leftsquigarrow \mathbf{normal}\ 2\ 3 \end{bmatrix} \llbracket \overline{x_2} + \overline{n} \rrbracket\ (\mathbf{snd}\ t)$$

$$\vdots$$

$$\vdots$$

$$\Longrightarrow \left(\left[\ell \leftsquigarrow \mathbf{normal}\ 0\ 1 \right],\ \begin{bmatrix} \overline{n} \leftsquigarrow \mathbf{return}\ \llbracket \ell \rrbracket; \\ \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (\mathbf{fst}\ t - \llbracket \ell \rrbracket)); \\ \overline{x_1} \leftsquigarrow \mathbf{return}\ (\mathbf{fst}\ t - \llbracket \ell \rrbracket); \\ \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (\mathbf{snd}\ t - \llbracket \ell \rrbracket)); \\ \overline{x_2} \leftsquigarrow \mathbf{return}\ (\mathbf{snd}\ t - \llbracket \ell \rrbracket); \\ \overline{p} \leftsquigarrow \mathbf{return}\ t \end{bmatrix} \right)$$

New input language: add literal arrays and indexing

$$
\begin{array}{lrl}
\text{Variables} & x & \\
\text{Real numbers} & r \in \mathbb{R} & \\
\text{Terms} & e ::= & x \\
 & \mid & r \mid -e \mid e + e \mid \mathbf{normalD}\ r\ r\ e \\
 & \mid & () \mid (e_1, e_2) \mid \mathbf{fst}\ e \mid \mathbf{snd}\ e \\
 & \mid & \langle e_1, e_2, \ldots, e_n \rangle \mid \mathbf{idx}\ e\ r \\
 & \mid & \mathbf{lebesgue} \mid \mathbf{normal}\ r\ r \\
 & \mid & \mathbf{return}\ e \mid \mathbf{do}\ \{x \leftsquigarrow e; e\} \mid \mathbf{do}\ \{\mathbf{factor}\ e; e\} \\
\text{Types} & \alpha, \beta ::= & \mathbb{1} \mid \mathbb{R} \mid \alpha \times \beta \mid \langle \alpha \rangle \mid \mathbb{M}\ \alpha
\end{array}
$$

A collection of five noisy measurements

2-D case

5-D case

$$
\begin{aligned}
m = \mathbf{do}\ \{ & n \leftsquigarrow \mathbf{normal}\ 0\ 1; \\
& p \leftsquigarrow \mathbf{do}\ \{ x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3; \\
& \qquad\qquad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3; \\
& \qquad\qquad \mathbf{return}\ (x_1 + n, x_2 + n)\}; \\
& \mathbf{return}\ (p, n)\}
\end{aligned}
$$

A collection of five noisy measurements

2-D case

$$
\begin{aligned}
m = \mathbf{do}\ \{& n \leftsquigarrow \mathbf{normal}\ 0\ 1\mathbf{;}\\
& p \leftsquigarrow \mathbf{do}\ \{ x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3\mathbf{;}\\
&\qquad\qquad\quad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3\mathbf{;}\\
&\qquad\qquad\quad \mathbf{return}\ (x_1 + n\,\mathbf{,}\, x_2 + n)\}\mathbf{;}\\
& \mathbf{return}\ (p\,\mathbf{,}\, n)\}
\end{aligned}
$$

5-D case

$$
\begin{aligned}
m = \mathbf{do}\ \{& n \leftsquigarrow \mathbf{normal}\ 0\ 1\mathbf{;}\\
& p \leftsquigarrow \mathbf{do}\ \{ x_0 \leftsquigarrow \mathbf{normal}\ 2\ 3\mathbf{;}\\
&\qquad\qquad\quad x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3\mathbf{;}\\
&\qquad\qquad\quad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3\mathbf{;}\\
&\qquad\qquad\quad x_3 \leftsquigarrow \mathbf{normal}\ 2\ 3\mathbf{;}\\
&\qquad\qquad\quad x_4 \leftsquigarrow \mathbf{normal}\ 2\ 3\mathbf{;}\\
&\qquad\qquad\quad \mathbf{return}\ \langle x_0 + n, x_1 + n, \ldots, x_4 + n\rangle\}\mathbf{;}\\
& \mathbf{return}\ (p\,\mathbf{,}\, n)\}
\end{aligned}
$$

A collection of five noisy measurements

2-D case

$$
\begin{aligned}
m = \mathbf{do}\ \{& n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
& p \leftsquigarrow \mathbf{do}\ \{ x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
& \qquad\qquad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
& \qquad\qquad \mathbf{return}\ (x_1 + n, x_2 + n)\};\\
& \mathbf{return}\ (p, n)\}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{disintegrate}\ m\ t =\\
&\mathbf{do}\ \{ \ell \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
&\quad n \leftsquigarrow \mathbf{return}\ [\![\ell]\!];\\
&\quad \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (\mathbf{fst}\ t - [\![\ell]\!]));\\
&\quad x_1 \leftsquigarrow \mathbf{return}\ (\mathbf{fst}\ t - [\![\ell]\!]);\\
&\quad \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (\mathbf{snd}\ t - [\![\ell]\!]));\\
&\quad x_2 \leftsquigarrow \mathbf{return}\ (\mathbf{snd}\ t - [\![\ell]\!]);\\
&\quad p \leftsquigarrow \mathbf{return}\ t;\\
&\quad \mathbf{return}\ n\}
\end{aligned}
$$

5-D case

$$
\begin{aligned}
m = \mathbf{do}\ \{& n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
& p \leftsquigarrow \mathbf{do}\ \{ x_0 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
& \qquad\qquad x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
& \qquad\qquad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
& \qquad\qquad x_3 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
& \qquad\qquad x_4 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
& \qquad\qquad \mathbf{return}\ \langle x_0 + n, x_1 + n, \ldots, x_4 + n\rangle\};\\
& \mathbf{return}\ (p, n)\}
\end{aligned}
$$

A collection of five noisy measurements

2-D case

$$
\begin{aligned}
m = \mathbf{do}\ \{&n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
&p \leftsquigarrow \mathbf{do}\ \{x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
&\qquad\qquad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
&\qquad\qquad \mathbf{return}\ (x_1 + n, x_2 + n)\};\\
&\mathbf{return}\ (p, n)\}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{disintegrate}\ m\ t =\\
&\mathbf{do}\ \{\ell \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
&\quad n \leftsquigarrow \mathbf{return}\ [\![\ell]\!];\\
&\quad \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (\mathbf{fst}\ t - [\![\ell]\!]));\\
&\quad x_1 \leftsquigarrow \mathbf{return}\ (\mathbf{fst}\ t - [\![\ell]\!]);\\
&\quad \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (\mathbf{snd}\ t - [\![\ell]\!]));\\
&\quad x_2 \leftsquigarrow \mathbf{return}\ (\mathbf{snd}\ t - [\![\ell]\!]);\\
&\quad p \leftsquigarrow \mathbf{return}\ t;\\
&\quad \mathbf{return}\ n\}
\end{aligned}
$$

5-D case

$$
\begin{aligned}
m = \mathbf{do}\ \{&n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
&p \leftsquigarrow \mathbf{do}\ \{x_0 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
&\qquad\qquad x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
&\qquad\qquad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
&\qquad\qquad x_3 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
&\qquad\qquad x_4 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
&\qquad\qquad \mathbf{return}\ \langle x_0 + n, x_1 + n, \ldots, x_4 + n\rangle\};\\
&\mathbf{return}\ (p, n)\}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{disintegrate}\ m\ t =\\
&\mathbf{do}\ \{\ell \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
&\quad n \leftsquigarrow \mathbf{return}\ [\![\ell]\!];\\
&\quad \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (\mathbf{idx}\ t\ 0 - [\![\ell]\!]));\\
&\quad x_0 \leftsquigarrow \mathbf{return}\ (\mathbf{idx}\ t\ 0 - [\![\ell]\!]);\\
&\quad \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (\mathbf{idx}\ t\ 1 - [\![\ell]\!]));\\
&\quad x_1 \leftsquigarrow \mathbf{return}\ (\mathbf{idx}\ t\ 1 - [\![\ell]\!]);\\
&\quad \vdots\\
&\quad \mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (\mathbf{idx}\ t\ 4 - [\![\ell]\!]));\\
&\quad x_4 \leftsquigarrow \mathbf{return}\ (\mathbf{idx}\ t\ 4 - [\![\ell]\!]);\\
&\quad p \leftsquigarrow \mathbf{return}\ t;\\
&\quad \mathbf{return}\ n\}
\end{aligned}
$$

Internal language: use heaps and locations as before

Variables	x
Locations	$\overline{x}$
Real numbers $r \in \mathbb{R}$	
Bindings	$b ::= \overline{x} \leftsquigarrow e \mid \textbf{factor}\ e$
Heap	$h ::= [b_1; b_2; \ldots; b_n]$
Terms	$e ::= x \mid \overline{x}$
	$\mid\ r \mid -e \mid e + e \mid \textbf{normalD}\ r\ r\ e$
	$\mid\ () \mid (e_1, e_2) \mid \textbf{fst}\ e \mid \textbf{snd}\ e$
	$\mid\ \langle e_1, e_2, \ldots, e_n \rangle \mid \textbf{idx}\ e\ r$
	$\mid\ \textbf{lebesgue} \mid \textbf{normal}\ r\ r$
	$\mid\ \textbf{return}\ e \mid \textbf{do}\ \{x \leftsquigarrow e; e\} \mid \textbf{do}\ \{\textbf{factor}\ e; e\}$
Types	$\alpha, \beta ::= \mathbb{1} \mid \mathbb{R} \mid \alpha \times \beta \mid \langle \alpha \rangle \mid \mathbb{M}\ \alpha$

Step 2: $\lhd\, h_1\ (fst\ v)\ t$

2-D case

5-D case

$$\lhd \begin{bmatrix} \overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1; \\ \overline{p} \leftsquigarrow \textbf{do}\ \{x_1 \leftsquigarrow \textbf{normal}\ 2\ 3; \\ \qquad\qquad x_2 \leftsquigarrow \textbf{normal}\ 2\ 3; \\ \qquad\qquad \textbf{return}\ (x_1 + \overline{n}, x_2 + \overline{n})\} \end{bmatrix} [\![\overline{p}]\!]\ t$$

$$\vdots$$

$$\lhd \left[\overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1; \overline{x_1} \leftsquigarrow \textbf{normal}\ 2\ 3; \overline{x_2} \leftsquigarrow \textbf{normal}\ 2\ 3\right] [\![(\overline{x_1} + \overline{n}, \overline{x_2} + \overline{n})]\!]\ t$$

$$\lhd \left[\overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1; \overline{x_1} \leftsquigarrow \textbf{normal}\ 2\ 3; \overline{x_2} \leftsquigarrow \textbf{normal}\ 2\ 3\right] [\![\overline{x_1} + \overline{n}]\!]\ (\textbf{fst}\ t)$$

$$\vdots$$

$$\lhd \begin{bmatrix} \overline{n} \leftsquigarrow \textbf{return}\ [\![\ell]\!]; \\ \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{fst}\ t - [\![\ell]\!])); \\ \overline{x_1} \leftsquigarrow \textbf{return}\ (\textbf{fst}\ t - [\![\ell]\!]); \\ \overline{x_2} \leftsquigarrow \textbf{normal}\ 2\ 3 \end{bmatrix} [\![\overline{x_2} + \overline{n}]\!]\ (\textbf{snd}\ t)$$

$$\vdots$$

$$\vdots$$

$$\Longrightarrow \left(\left[\ell \leftsquigarrow \textbf{normal}\ 0\ 1\right], \begin{bmatrix} \overline{n} \leftsquigarrow \textbf{return}\ [\![\ell]\!]; \\ \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{fst}\ t - [\![\ell]\!])); \\ \overline{x_1} \leftsquigarrow \textbf{return}\ (\textbf{fst}\ t - [\![\ell]\!]); \\ \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{snd}\ t - [\![\ell]\!])); \\ \overline{x_2} \leftsquigarrow \textbf{return}\ (\textbf{snd}\ t - [\![\ell]\!]); \\ \overline{p} \leftsquigarrow \textbf{return}\ t \end{bmatrix} \right)$$

Step 2: $\triangleleft\ h_1\ (fst\ v)\ t$

2-D case

$$\triangleleft \left[\begin{array}{l} \overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1; \\ \overline{p} \leftsquigarrow \textbf{do}\ \{x_1 \leftsquigarrow \textbf{normal}\ 2\ 3; \\ \qquad\quad x_2 \leftsquigarrow \textbf{normal}\ 2\ 3; \\ \qquad\quad \textbf{return}\ (x_1 + \overline{n}, x_2 + \overline{n})\} \end{array}\right] \llbracket \overline{p} \rrbracket\ t$$

$$\vdots$$

$$\triangleleft \left[\overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1 ; \overline{x_1} \leftsquigarrow \textbf{normal}\ 2\ 3 ; \overline{x_2} \leftsquigarrow \textbf{normal}\ 2\ 3\right] \llbracket (\overline{x_1} + \overline{n}, \overline{x_2} + \overline{n}) \rrbracket\ t$$

$$\triangleleft \left[\overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1 ; \overline{x_1} \leftsquigarrow \textbf{normal}\ 2\ 3 ; \overline{x_2} \leftsquigarrow \textbf{normal}\ 2\ 3\right] \llbracket \overline{x_1} + \overline{n} \rrbracket\ (\textbf{fst}\ t)$$

$$\vdots$$

$$\triangleleft \left[\begin{array}{l} \overline{n} \leftsquigarrow \textbf{return}\ \llbracket \ell \rrbracket; \\ \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{fst}\ t - \llbracket \ell \rrbracket)); \\ \overline{x_1} \leftsquigarrow \textbf{return}\ (\textbf{fst}\ t - \llbracket \ell \rrbracket); \\ \overline{x_2} \leftsquigarrow \textbf{normal}\ 2\ 3 \end{array}\right] \llbracket \overline{x_2} + \overline{n} \rrbracket\ (\textbf{snd}\ t)$$

$$\vdots$$

$$\vdots$$

$$\Longrightarrow \left(\left[\ell \leftsquigarrow \textbf{normal}\ 0\ 1\right], \left[\begin{array}{l} \overline{n} \leftsquigarrow \textbf{return}\ \llbracket \ell \rrbracket; \\ \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{fst}\ t - \llbracket \ell \rrbracket)); \\ \overline{x_1} \leftsquigarrow \textbf{return}\ (\textbf{fst}\ t - \llbracket \ell \rrbracket); \\ \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{snd}\ t - \llbracket \ell \rrbracket)); \\ \overline{x_2} \leftsquigarrow \textbf{return}\ (\textbf{snd}\ t - \llbracket \ell \rrbracket); \\ \overline{p} \leftsquigarrow \textbf{return}\ t \end{array}\right] \right)$$

5-D case

$$\triangleleft \left[\begin{array}{l} \overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1; \\ \overline{p} \leftsquigarrow \textbf{do}\ \{x_0 \leftsquigarrow \textbf{normal}\ 2\ 3; \\ \qquad\quad x_1 \leftsquigarrow \textbf{normal}\ 2\ 3; \\ \qquad\quad \vdots \\ \qquad\quad x_4 \leftsquigarrow \textbf{normal}\ 2\ 3; \\ \qquad\quad \textbf{return}\ \langle x_0 + \overline{n}, x_1 + \overline{n}, \ldots, x_4 + \overline{n} \rangle\} \end{array}\right] \llbracket \overline{p} \rrbracket\ t$$

$$\vdots$$

$$\triangleleft \left[\overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1 ; \overline{x_0} \leftsquigarrow \textbf{normal}\ 2\ 3 ; \overline{x_1} \leftsquigarrow \textbf{normal}\ 2\ 3 ; \ldots\right] \llbracket \langle \overline{x_0} + \overline{n}, \overline{x_1} + \overline{n}, \ldots, \overline{x_4} + \overline{n} \rangle \rrbracket\ t$$

$$\triangleleft \left[\overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1 ; \overline{x_0} \leftsquigarrow \textbf{normal}\ 2\ 3 ; \ldots\right] \llbracket \overline{x_0} + \overline{n} \rrbracket\ (\textbf{idx}\ t\ 0)$$

$$\vdots$$

$$\triangleleft \left[\ldots ; \overline{x_0} \leftsquigarrow \textbf{return}\ (\textbf{idx}\ t\ 0 - \llbracket \ell \rrbracket) ; \overline{x_1} \leftsquigarrow \textbf{normal}\ 2\ 3 ; \ldots\right] \llbracket \overline{x_1} + \overline{n} \rrbracket\ (\textbf{idx}\ t\ 1)$$

$$\vdots$$

$$\vdots$$

$$\triangleleft \left[\ldots ; \overline{x_1} \leftsquigarrow \textbf{return}\ (\textbf{idx}\ t\ 1 - \llbracket \ell \rrbracket) ; \ldots ; \overline{x_4} \leftsquigarrow \textbf{normal}\ 2\ 3\right] \llbracket \overline{x_4} + \overline{n} \rrbracket\ (\textbf{idx}\ t\ 4)$$

$$\vdots$$

$$\vdots$$

$$\Longrightarrow \left(\left[\ell \leftsquigarrow \textbf{normal}\ 0\ 1\right], \left[\begin{array}{l} \overline{n} \leftsquigarrow \textbf{return}\ \llbracket \ell \rrbracket; \\ \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{idx}\ t\ 0 - \llbracket \ell \rrbracket)); \\ \overline{x_0} \leftsquigarrow \textbf{return}\ (\textbf{idx}\ t\ 0 - \llbracket \ell \rrbracket); \\ \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{idx}\ t\ 1 - \llbracket \ell \rrbracket)); \\ \overline{x_1} \leftsquigarrow \textbf{return}\ (\textbf{idx}\ t\ 1 - \llbracket \ell \rrbracket); \\ \vdots \\ \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{idx}\ t\ 4 - \llbracket \ell \rrbracket)); \\ \overline{x_4} \leftsquigarrow \textbf{return}\ (\textbf{idx}\ t\ 4 - \llbracket \ell \rrbracket); \\ \overline{p} \leftsquigarrow \textbf{return}\ t \end{array}\right] \right)$$

For large arrays we could use **plate**

Say **plate** is a macro:

For large arrays we could use **plate**

Say **plate** is a macro:

$$
\begin{array}{l}
\textbf{plate}\ n\ i\ m = \\
\textbf{do}\ \{ x_0 \leftarrow\!\!\sim m\{i \mapsto 0\}\textbf{;} \\
\qquad x_1 \leftarrow\!\!\sim m\{i \mapsto 1\}\textbf{;} \\
\qquad\ \ \vdots \\
\qquad x_{n-1} \leftarrow\!\!\sim m\{i \mapsto n-1\}\textbf{;} \\
\qquad \textbf{return}\ \langle x_0, x_1, \ldots, x_{n-1} \rangle \}
\end{array}
$$

For large arrays we could use **plate**

Say **plate** is a macro:

$$\begin{aligned}
&\textbf{plate}\ n\ i\ m = \\
&\textbf{do}\ \{x_0 \leftsquigarrow m\{i \mapsto 0\}; \\
&\qquad x_1 \leftsquigarrow m\{i \mapsto 1\}; \\
&\qquad\ \vdots \\
&\qquad x_{n-1} \leftsquigarrow m\{i \mapsto n-1\}; \\
&\qquad \textbf{return}\ \langle x_0, x_1, \ldots, x_{n-1}\rangle\}
\end{aligned}$$

We could express a hundred noisy measurements as:

$$\begin{aligned}
&\textbf{do}\ \{n \leftsquigarrow \textbf{normal}\ 0\ 1; \\
&\qquad p \leftsquigarrow \textbf{plate}\ 100\ _\ (\textbf{do}\ \{x \leftsquigarrow \textbf{normal}\ 2\ 3; \\
&\qquad\qquad\qquad\qquad\qquad\qquad \textbf{return}\ (x + n)\}); \\
&\qquad \textbf{return}\ (p, n)\}
\end{aligned}$$

For large arrays we could use **plate**

Say **plate** is a macro:

$$
\begin{aligned}
&\textbf{plate}\ n\ i\ m =\\
&\textbf{do}\ \{x_0 \leftsquigarrow m\{i \mapsto 0\};\\
&\qquad x_1 \leftsquigarrow m\{i \mapsto 1\};\\
&\qquad\ \vdots\\
&\qquad x_{n-1} \leftsquigarrow m\{i \mapsto n-1\};\\
&\qquad \textbf{return}\ \langle x_0, x_1, \ldots, x_{n-1}\rangle\}
\end{aligned}
$$

We could express a hundred noisy measurements as:

$$
\begin{aligned}
&\textbf{do}\ \{n \leftsquigarrow \textbf{normal}\ 0\ 1;\\
&\qquad p \leftsquigarrow \textbf{plate}\ 100\ \_\ (\textbf{do}\ \{x \leftsquigarrow \textbf{normal}\ 2\ 3;\\
&\qquad\qquad\qquad\qquad\qquad\quad \textbf{return}\ (x+n)\});\\
&\qquad \textbf{return}\ (p, n)\}
\end{aligned}
$$

While unrolling is correct in principle, we could do better with an abstract approach.

Input language: add array and plate as primitives

$$
\begin{array}{lrl}
\text{Variables} & x & \\
\text{Real numbers} & r \in \mathbb{R} & \\
\text{Terms} & e ::= & x \\
& \mid & r \mid \mathbf{-}e \mid e \mathbf{+} e \mid \mathbf{normalD}\ r\ r\ e \\
& \mid & \mathbf{()} \mid \mathbf{(}e_1\mathbf{,}e_2\mathbf{)} \mid \mathbf{fst}\ e \mid \mathbf{snd}\ e \\
& \mid & \color{red}{\mathbf{array}\ e\ x\ e} \mid \color{red}{\mathbf{plate}\ e\ x\ e} \mid \color{red}{\mathbf{idx}\ e\ x} \\
& \mid & \mathbf{lebesgue} \mid \mathbf{normal}\ r\ r \\
& \mid & \mathbf{return}\ e \mid \mathbf{do}\ \{x \leftsquigarrow e\mathbf{;}\ e\} \mid \mathbf{do}\ \{\mathbf{factor}\ e\mathbf{;}\ e\} \\
\text{Types} & \alpha, \beta ::= & \mathbb{1} \mid \mathbb{R} \mid \alpha \times \beta \mid \langle \alpha \rangle \mid \mathbb{M}\ \alpha
\end{array}
$$

A collection of many noisy measurements

5-D case

k-D case

$$
\begin{aligned}
m = \textbf{do}\ \{ & n \leftsquigarrow \textbf{normal}\ 0\ 1; \\
& p \leftsquigarrow \textbf{do}\ \{ x_0 \leftsquigarrow \textbf{normal}\ 2\ 3; \\
& \qquad\qquad x_1 \leftsquigarrow \textbf{normal}\ 2\ 3; \\
& \qquad\qquad x_2 \leftsquigarrow \textbf{normal}\ 2\ 3; \\
& \qquad\qquad x_3 \leftsquigarrow \textbf{normal}\ 2\ 3; \\
& \qquad\qquad x_4 \leftsquigarrow \textbf{normal}\ 2\ 3; \\
& \qquad\qquad \textbf{return}\ \langle x_0 + n, x_1 + n, \ldots, x_4 + n \rangle \}; \\
& \textbf{return}\ (p, n) \}
\end{aligned}
$$

A collection of many noisy measurements

5-D case

$$
\begin{aligned}
m = \mathbf{do}\ \{&n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
&p \leftsquigarrow \mathbf{do}\ \{x_0 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
&\qquad\qquad x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
&\qquad\qquad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
&\qquad\qquad x_3 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
&\qquad\qquad x_4 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
&\qquad\qquad \mathbf{return}\ \langle x_0 + n, x_1 + n, \ldots, x_4 + n\rangle\};\\
&\mathbf{return}\ (p, n)\}
\end{aligned}
$$

k-D case

$$
\begin{aligned}
m = \mathbf{do}\ \{&n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
&p \leftsquigarrow \mathbf{plate}\ k\ i\ (\mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
&\qquad\qquad\qquad\qquad \mathbf{return}\ (x + n)\});\\
&\mathbf{return}\ (p, n)\}
\end{aligned}
$$

A collection of many noisy measurements

5-D case

$$
\begin{aligned}
m = \textbf{do}\ \{& n \leftsquigarrow \textbf{normal}\ 0\ 1; \\
& p \leftsquigarrow \textbf{do}\ \{ x_0 \leftsquigarrow \textbf{normal}\ 2\ 3; \\
& \qquad\qquad x_1 \leftsquigarrow \textbf{normal}\ 2\ 3; \\
& \qquad\qquad x_2 \leftsquigarrow \textbf{normal}\ 2\ 3; \\
& \qquad\qquad x_3 \leftsquigarrow \textbf{normal}\ 2\ 3; \\
& \qquad\qquad x_4 \leftsquigarrow \textbf{normal}\ 2\ 3; \\
& \qquad\qquad \textbf{return}\ \langle x_0 + n, x_1 + n, \ldots, x_4 + n \rangle \}; \\
& \textbf{return}\ (p, n) \}
\end{aligned}
$$

$$
\begin{aligned}
& \textbf{disintegrate}\ m\ t = \\
& \textbf{do}\ \{ \ell \leftsquigarrow \textbf{normal}\ 0\ 1; \\
& \quad n \leftsquigarrow \textbf{return}\ [\![\ell]\!]; \\
& \quad \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{idx}\ t\ 0 - [\![\ell]\!])); \\
& \quad x_0 \leftsquigarrow \textbf{return}\ (\textbf{idx}\ t\ 0 - [\![\ell]\!]); \\
& \quad \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{idx}\ t\ 1 - [\![\ell]\!])); \\
& \quad x_1 \leftsquigarrow \textbf{return}\ (\textbf{idx}\ t\ 1 - [\![\ell]\!]); \\
& \quad \vdots \\
& \quad \textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{idx}\ t\ 4 - [\![\ell]\!])); \\
& \quad x_4 \leftsquigarrow \textbf{return}\ (\textbf{idx}\ t\ 4 - [\![\ell]\!]); \\
& \quad p \leftsquigarrow \textbf{return}\ t; \\
& \quad \textbf{return}\ n \}
\end{aligned}
$$

k-D case

$$
\begin{aligned}
m = \textbf{do}\ \{& n \leftsquigarrow \textbf{normal}\ 0\ 1; \\
& p \leftsquigarrow \textbf{plate}\ k\ i\ (\textbf{do}\ \{ x \leftsquigarrow \textbf{normal}\ 2\ 3; \\
& \qquad\qquad\qquad\qquad \textbf{return}\ (x + n) \}); \\
& \textbf{return}\ (p, n) \}
\end{aligned}
$$

A collection of many noisy measurements

5-D case

```
m = do {n ⇜ normal 0 1;
        p ⇜ do {x₀ ⇜ normal 2 3;
                x₁ ⇜ normal 2 3;
                x₂ ⇜ normal 2 3;
                x₃ ⇜ normal 2 3;
                x₄ ⇜ normal 2 3;
                return ⟨x₀ + n, x₁ + n, . . . , x₄ + n⟩};
        return (p , n)}
```

```
disintegrate m t =
do {ℓ ⇜ normal 0 1;
    n ⇜ return ⟦ℓ⟧;
    factor (normalD 2 3 (idx t 0 − ⟦ℓ⟧));
    x₀ ⇜ return (idx t 0 − ⟦ℓ⟧);
    factor (normalD 2 3 (idx t 1 − ⟦ℓ⟧));
    x₁ ⇜ return (idx t 1 − ⟦ℓ⟧);
    ⋮
    factor (normalD 2 3 (idx t 4 − ⟦ℓ⟧));
    x₄ ⇜ return (idx t 4 − ⟦ℓ⟧);
    p ⇜ return t;
    return n}
```

k-D case

```
m = do {n ⇜ normal 0 1;
        p ⇜ plate k i (do {x ⇜ normal 2 3;
                           return (x + n)});
        return (p , n)}
```

```
disintegrate m t =
do {ℓ ⇜ normal 0 1;
    n ⇜ return ⟦ℓ⟧;
    _ ⇜ plate k i (factor (normalD 2 3 (idx t î − ⟦ℓ⟧)));
    x ⇜ return (idx t î − ⟦ℓ⟧);
    p ⇜ return t;
    return n}
```

Internal language: use indices and multilocs

Variables	x
Locations	$\overline{x}$
Indices	$\hat{x}$
Real numbers $r \in \mathbb{R}$	
Bindings	$b ::= \overline{x}_{[\hat{x},\ldots,\hat{y}]} \leftsquigarrow e \mid \textbf{factor}_{[\hat{x},\ldots,\hat{y}]}\ e$
Heap	$h ::= [b_1; b_2; \ldots; b_n]$
Terms	$e ::= x \mid \overline{x}_{[\hat{x},\ldots,\hat{y}]} \mid \langle\overline{x}\rangle_{[\hat{x},\ldots,\hat{y}]} \mid \hat{x}$
	$\mid\ r \mid -e \mid e + e \mid \textbf{normalD}\ r\ r\ e$
	$\mid\ () \mid (e_1, e_2) \mid \textbf{fst}\ e \mid \textbf{snd}\ e$
	$\mid\ \textbf{array}\ e\ x\ e \mid \textbf{plate}\ e\ x\ e \mid \textbf{idx}\ e\ x$
	$\mid\ \textbf{lebesgue} \mid \textbf{normal}\ r\ r$
	$\mid\ \textbf{return}\ e \mid \textbf{do}\ \{x \leftsquigarrow e; e\} \mid \textbf{do}\ \{\textbf{factor}\ e; e\}$
Types	$\alpha, \beta ::= \mathbb{1} \mid \mathbb{R} \mid \mathbb{I} \mid \alpha \times \beta \mid \langle\alpha\rangle \mid \mathbb{M}\ \alpha$

Now the four functions each take a set of indices

$$\ggg \text{ ("perform")} : \mathit{heap} \to \mathit{inds} \to \lceil \mathbb{M}\ \alpha \rceil \to (\mathit{emission} \times \mathit{heap} \times \lfloor \alpha \rfloor)$$

$$\triangleright \text{ ("evaluate")} : \mathit{heap} \to \mathit{inds} \to \lceil \alpha \rceil \to (\mathit{emission} \times \mathit{heap} \times \lfloor \alpha \rfloor)$$

$$\lll \text{ ("constrain outcome")} : \mathit{heap} \to \mathit{inds} \to \lceil \mathbb{M}\ \alpha \rceil \to \lfloor \alpha \rfloor \to (\mathit{emission} \times \mathit{heap})$$

$$\triangleleft \text{ ("constrain value")} : \mathit{heap} \to \mathit{inds} \to \lceil \alpha \rceil \to \lfloor \alpha \rfloor \to (\mathit{emission} \times \mathit{heap})$$

Step 1: $\rhd\ [\,]\ m$

5-D case

k-D case

$$
\left[\begin{array}{l}
\rhd\ [\,]\ [\![\mathbf{do}\ \{n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
\quad p \leftsquigarrow \mathbf{do}\ \{x_0 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad x_3 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad x_4 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad \mathbf{return}\ \langle x_0+n, x_1+n, \ldots, x_4+n\rangle\};\\
\quad \mathbf{return}\ (p, n)\}]\!]\\
\quad\vdots\\
\Longrightarrow \left(\ [\,]\ ,\ \left[\begin{array}{l}
\overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
\overline{p} \leftsquigarrow \mathbf{do}\ \{x_0 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad x_3 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad x_4 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad \mathbf{return}\ \langle x_0+n, x_1+n, \ldots, x_4+n\rangle\}
\end{array}\right],\ [\![(\overline{p}, \overline{n})]\!]\right)
\end{array}\right.
$$

Step 1: $\triangleright\!\!\triangleright\ [\,]\ m$

5-D case

$$
\left[\begin{array}{l}
\triangleright\!\!\triangleright\ [\,]\ [\![\mathbf{do}\ \{n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
\qquad\qquad p \leftsquigarrow \mathbf{do}\ \{x_0 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad\qquad\qquad x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad\qquad\qquad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad\qquad\qquad x_3 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad\qquad\qquad x_4 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad\qquad\qquad \mathbf{return}\ \langle x_0 + n, x_1 + n, \ldots, x_4 + n\rangle\};\\
\qquad\qquad \mathbf{return}\ (\,p\,,n\,)\}]\!]\\
\qquad \vdots\\
\Longrightarrow \left([\,]\ ,\ \left[\begin{array}{l}
\overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
\overline{p} \leftsquigarrow \mathbf{do}\ \{x_0 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad x_2 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad x_3 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad x_4 \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad \mathbf{return}\ \langle x_0 + n, x_1 + n, \ldots, x_4 + n\rangle\}
\end{array}\right],\ [\![(\,\overline{p}\,,\overline{n}\,)]\!] \right)
\end{array}\right.
$$

k-D case

$$
\left[\begin{array}{l}
\triangleright\!\!\triangleright\ [\,]\ [\,]\ [\![\mathbf{do}\ \{n \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
\qquad\qquad p \leftsquigarrow \mathbf{plate}\ k\ i\ (\mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad\qquad\qquad\qquad\qquad \mathbf{return}\ (x + n)\});\\
\qquad\qquad \mathbf{return}\ (\,p\,,n\,)\}]\!]\\
\qquad \vdots\\
\Longrightarrow \left([\,]\ ,\ \left[\begin{array}{l}
\overline{n}_{[\,]} \leftsquigarrow \mathbf{normal}\ 0\ 1;\\
\overline{p}_{[\,]} \leftsquigarrow \mathbf{plate}\ k\ i\ (\mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3;\\
\qquad\qquad\qquad\qquad \mathbf{return}\ (x + \overline{n})\});
\end{array}\right],\ [\![(\,\overline{p}\,,\overline{n}\,)]\!] \right)
\end{array}\right.
$$

Step 2: $\triangleleft\ h_1\ (fst\ v)\ t$

5-D case

k-D case

$$
\triangleleft \left[\begin{array}{l}
\overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1;\\
\overline{p} \leftsquigarrow \textbf{do}\ \{x_0 \leftsquigarrow \textbf{normal}\ 2\ 3;\\
\qquad x_1 \leftsquigarrow \textbf{normal}\ 2\ 3;\\
\qquad \vdots\\
\qquad x_4 \leftsquigarrow \textbf{normal}\ 2\ 3;\\
\qquad \textbf{return}\ \langle x_0 + \overline{n}, x_1 + \overline{n}, \ldots, x_4 + \overline{n}\rangle\}
\end{array}\right] \llbracket \overline{p} \rrbracket\ t
$$

$$\vdots$$

$$\triangleleft\ [\overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1\,; \overline{x_0} \leftsquigarrow \textbf{normal}\ 2\ 3\,; \overline{x_1} \leftsquigarrow \textbf{normal}\ 2\ 3\,; \ldots]\ \llbracket \langle \overline{x_0} + \overline{n}, \overline{x_1} + \overline{n}, \ldots, \overline{x_4} + \overline{n} \rangle \rrbracket\ t$$

$$\triangleleft\ [\overline{n} \leftsquigarrow \textbf{normal}\ 0\ 1\,; \overline{x_0} \leftsquigarrow \textbf{normal}\ 2\ 3\,; \ldots]\ \llbracket \overline{x_0} + \overline{n} \rrbracket\ (\textbf{idx}\ t\ 0)$$

$$\vdots$$

$$\triangleleft\ [\ldots; \overline{x_0} \leftsquigarrow \textbf{return}\ (\textbf{idx}\ t\ 0 - \llbracket \ell \rrbracket)\,; \overline{x_1} \leftsquigarrow \textbf{normal}\ 2\ 3\,; \ldots]\ \llbracket \overline{x_1} + \overline{n} \rrbracket\ (\textbf{idx}\ t\ 1)$$

$$\vdots$$

$$\vdots$$

$$\triangleleft\ [\ldots; \overline{x_1} \leftsquigarrow \textbf{return}\ (\textbf{idx}\ t\ 1 - \llbracket \ell \rrbracket)\,; \ldots; \overline{x_4} \leftsquigarrow \textbf{normal}\ 2\ 3]\ \llbracket \overline{x_4} + \overline{n} \rrbracket\ (\textbf{idx}\ t\ 4)$$

$$\vdots$$

$$\vdots$$

$$
\Longrightarrow \left([\ell \leftsquigarrow \textbf{normal}\ 0\ 1]\ , \left[\begin{array}{l}
\overline{n} \leftsquigarrow \textbf{return}\ \llbracket \ell \rrbracket;\\
\textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{idx}\ t\ 0 - \llbracket \ell \rrbracket));\\
\overline{x_0} \leftsquigarrow \textbf{return}\ (\textbf{idx}\ t\ 0 - \llbracket \ell \rrbracket);\\
\textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{idx}\ t\ 1 - \llbracket \ell \rrbracket));\\
\overline{x_1} \leftsquigarrow \textbf{return}\ (\textbf{idx}\ t\ 1 - \llbracket \ell \rrbracket);\\
\vdots\\
\textbf{factor}\ (\textbf{normalD}\ 2\ 3\ (\textbf{idx}\ t\ 4 - \llbracket \ell \rrbracket));\\
\overline{x_4} \leftsquigarrow \textbf{return}\ (\textbf{idx}\ t\ 4 - \llbracket \ell \rrbracket);\\
\overline{p} \leftsquigarrow \textbf{return}\ t
\end{array}\right] \right)
$$

Step 2: $\triangleleft\ h_1\ (\mathit{fst}\ v)\ t$

5-D case

$$
\left[\begin{array}{l}
\triangleleft \left[\begin{array}{l}
\overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1; \\
\overline{p} \leftsquigarrow \mathbf{do}\ \{x_0 \leftsquigarrow \mathbf{normal}\ 2\ 3; \\
\qquad\quad x_1 \leftsquigarrow \mathbf{normal}\ 2\ 3; \\
\qquad\quad \vdots \\
\qquad\quad x_4 \leftsquigarrow \mathbf{normal}\ 2\ 3; \\
\qquad\quad \mathbf{return}\ \langle x_0 + \overline{n}, x_1 + \overline{n}, \ldots, x_4 + \overline{n} \rangle \}
\end{array}\right] \llbracket \overline{p} \rrbracket\ t \\
\left[\begin{array}{l}
\vdots \\
\triangleleft\ [\overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1\,;\overline{x_0} \leftsquigarrow \mathbf{normal}\ 2\ 3\,;\overline{x_1} \leftsquigarrow \mathbf{normal}\ 2\ 3\,;\ldots]\ \llbracket \langle \overline{x_0} + \overline{n}, \overline{x_1} + \overline{n}, \ldots, \overline{x_4} + \overline{n} \rangle \rrbracket\ t \\
\left[\begin{array}{l}
\triangleleft\ [\overline{n} \leftsquigarrow \mathbf{normal}\ 0\ 1\,;\overline{x_0} \leftsquigarrow \mathbf{normal}\ 2\ 3\,;\ldots]\ \llbracket \overline{x_0} + \overline{n} \rrbracket\ (\mathbf{idx}\ t\ 0) \\
\vdots
\end{array}\right. \\
\left[\begin{array}{l}
\triangleleft\ [\ldots;\overline{x_0} \leftsquigarrow \mathbf{return}\ (\mathbf{idx}\ t\ 0 - \llbracket \ell \rrbracket)\,;\overline{x_1} \leftsquigarrow \mathbf{normal}\ 2\ 3\,;\ldots]\ \llbracket \overline{x_1} + \overline{n} \rrbracket\ (\mathbf{idx}\ t\ 1) \\
\vdots
\end{array}\right. \\
\vdots \\
\left[\begin{array}{l}
\triangleleft\ [\ldots;\overline{x_1} \leftsquigarrow \mathbf{return}\ (\mathbf{idx}\ t\ 1 - \llbracket \ell \rrbracket)\,;\ldots;\overline{x_4} \leftsquigarrow \mathbf{normal}\ 2\ 3]\ \llbracket \overline{x_4} + \overline{n} \rrbracket\ (\mathbf{idx}\ t\ 4) \\
\vdots
\end{array}\right. \\
\vdots
\end{array}\right. \\
\Longrightarrow \left([\ell \leftsquigarrow \mathbf{normal}\ 0\ 1]\ ,\ \left[\begin{array}{l}
\overline{n} \leftsquigarrow \mathbf{return}\ \llbracket \ell \rrbracket; \\
\mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (\mathbf{idx}\ t\ 0 - \llbracket \ell \rrbracket)); \\
\overline{x_0} \leftsquigarrow \mathbf{return}\ (\mathbf{idx}\ t\ 0 - \llbracket \ell \rrbracket); \\
\mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (\mathbf{idx}\ t\ 1 - \llbracket \ell \rrbracket)); \\
\overline{x_1} \leftsquigarrow \mathbf{return}\ (\mathbf{idx}\ t\ 1 - \llbracket \ell \rrbracket); \\
\vdots \\
\mathbf{factor}\ (\mathbf{normalD}\ 2\ 3\ (\mathbf{idx}\ t\ 4 - \llbracket \ell \rrbracket)); \\
\overline{x_4} \leftsquigarrow \mathbf{return}\ (\mathbf{idx}\ t\ 4 - \llbracket \ell \rrbracket); \\
\overline{p} \leftsquigarrow \mathbf{return}\ t
\end{array}\right] \right)
\end{array}\right.
$$

k-D case

$$
\left[\begin{array}{l}
\triangleleft \left[\begin{array}{l}
\overline{n}_{[]} \leftsquigarrow \mathbf{normal}\ 0\ 1; \\
\overline{p}_{[]} \leftsquigarrow \mathbf{plate}\ k\ i\ (\mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3; \\
\qquad\qquad\qquad\qquad\quad \mathbf{return}\ (x + \overline{n})\})
\end{array}\right]\ []\ \llbracket \overline{p} \rrbracket\ t \\
\left[\begin{array}{llll}
\lll\ [\overline{n}_{[]} \leftsquigarrow \mathbf{normal}\ 0\ 1] & [] & \llbracket \mathbf{plate}\ k\ i\ (\mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3; \ \mathbf{return}\ (x + \overline{n})\})\rrbracket & t \\
\triangleleft \left[\begin{array}{l}
\overline{n}_{[]} \leftsquigarrow \mathbf{normal}\ 0\ 1; \\
\overline{p}_{[\hat{i}]} \leftsquigarrow \mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3; \\
\qquad\qquad \mathbf{return}\ (x + \overline{n})\}
\end{array}\right] & [] & \llbracket \langle \overline{p} \rangle_{[]} \rrbracket & t \\
\lll\ [\overline{n}_{[]} \leftsquigarrow \mathbf{normal}\ 0\ 1] & [\hat{i}] & \llbracket \mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 2\ 3; \ \mathbf{return}\ (x + \overline{n})\}\rrbracket & (\mathbf{idx}\ t\ \hat{i}) \\
 & & \vdots & \\
\triangleleft \left[\begin{array}{l}
\overline{n}_{[]} \leftsquigarrow \mathbf{normal}\ 0\ 1; \\
\overline{x}_{[\hat{i}]} \leftsquigarrow \mathbf{normal}\ 2\ 3
\end{array}\right] & [\hat{i}] & \llbracket \overline{x} + \overline{n} \rrbracket & (\mathbf{idx}\ t\ \hat{i}) \\
 & & \vdots &
\end{array}\right. \\
\Longrightarrow \left([\ell \leftsquigarrow \mathbf{normal}\ 0\ 1]\ ,\ \left[\begin{array}{l}
\overline{n}_{[]} \leftsquigarrow \mathbf{return}\ \llbracket \ell \rrbracket; \\
\mathbf{factor}_{[\hat{i}]}\ (\mathbf{normalD}\ 2\ 3\ (\mathbf{idx}\ t\ \hat{i} - \llbracket \ell \rrbracket)); \\
\overline{x}_{[\hat{i}]} \leftsquigarrow \mathbf{return}\ (\mathbf{idx}\ t\ \hat{i} - \llbracket \ell \rrbracket); \\
\overline{p}_{[]} \leftsquigarrow \mathbf{return}\ t
\end{array}\right] \right)
\end{array}\right.
$$

Summary

Handle conditioning via *disintegration*

Summary

Handle conditioning via *disintegration*

Implement disintegration via *lazy partial evaluation*

Summary

Handle conditioning via *disintegration*

Implement disintegration via *lazy partial evaluation*

Generalize disintegration to condition *arrays*

Thank you

https://github.com/hakaru-dev/hakaru